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ABSTRACT

The primary objectives of this paper are pedagogical: to provide a reliable semi-subjective transformation procedure that might be used without difficulty by beginning students in factor analysis; to clarify and extend the existing knowledge of oblique transformations in general; and to provide a brief but meaningful explication of the general obliquimax. Implicit in these three objectives is the fourth objective of presenting a paper that might be profitable for both the beginning student and the factor analytic theoretician. The first section, primarily background, discusses one of Thurstone's methods of determining oblique transformations. The second section discusses certain theoretical aspects of the general obliquimax to provide a basis for the development and understanding of the simplified obliquimax. Finally, the semi-subjective simplified obliquimax transformation is developed and discussed within the context of Thurstone's box problem. (Author/AE)

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THE SIMPLIFIED OBLIQUIMAX AS A MODIFICATION OF THURSTONE'S METHOD OF OBLIQUE TRANSFORMATION: ITS METHODOLOGY, PROPERTIES AND GENERAL NATURE

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Once a factor analyst has determined a factor space for a complex of variables he usually desires a basis for interpreting the factor space. Traditionally the basis for interpreting a factor space has been made relative to either orthogonal (uncorrelated) factor axes or oblique (correlated) factor axes. That is to say, the "loadings" of the initial factor loading matrix are transformed through the use of either an orthogonal or oblique transformation procedure.

Any oblique transformation solution encompasses three matrices. The general names given to these three matrices are the pattern matrix, the structure matrix and the factor intercorrelation matrix. The entries of a structure matrix represent the perpendicular projections of the variable vectors onto the oblique factor axes and with an appropriate scaling are by rows the correlations of the variables with the factors. The entries of a pattern matrix represent the parallel projections of the variable vectors onto the oblique factor axes and are by rows the standardized regression weights of a regression equation describing each of the observed variables in terms of the correlated factors. The entries of the factor intercorrelation matrix are just the correlations between the factors. Although most papers on oblique transformations do not deal specifically with orthogonal factor axes, one may regard an orthogonal transformation solution within the more general framework of oblique solutions. In the general oblique framework the orthogonal transformation solution might be thought of as a special solution in which the factor intercorrelation matrix is an identity matrix. When the factors are uncorrelated the parallel projections and perpendicular projections of the variable vectors are identical, thereby resulting in a structure matrix that is identical to a pattern matrix.

There have been in the past two schools of thought pertaining to the type of interpretation that should be made for an oblique solution. One mode of thought was based upon the work of Louis Thurstone (1947) while the other was based upon the work of Karl Holzinger (Holzinger and Harman, 1941). The Thurstone-Holzinger differences stem philosophically from the idea of invariance and geometrically from the definitions of factor axes used in obtaining the final solution matrices, which implicitly determine whether it is the pattern or structure matrix that should be used for the final interpretation. The Thurstone (1947) school of thought bases the final interpretation of the oblique transformation on a structure matrix while the Holzinger school (Holzinger and Harman, 1941) uses a pattern matrix for the final interpretation.

Holzinger defined his solutions using primary vectors, those vectors formed by the intersection of hyperplanes. Thus, Holzinger's solution matrices for interpretation were the primary pattern and the primary intercorrelation matrices. The loadings of the primary pattern matrix are defined geometrically as the parallel projections of the variable vectors onto the unit length primary vectors.

Thurstone (1947) defined his solutions using reference vectors, those vectors defined as normals to hyperplanes. Thurstone was concerned with the perpendicular projections of the variable vectors onto the unit length reference vectors. Although Thurstone was interested in the reference structure matrix, it is interesting to note that he usually reported the primary intercorrelation matrix along with the reference intercorrelation matrix.

Typically a solution matrix is desired which will have scientific meaning and interpretability. Scientific meaning and interpretability are facilitated when some of the entries of the solution matrix are very high and the remaining entries are zero or near zero. A zero entry in a solution matrix may be thought of as a vanishing projection, thus the objective in obtaining a solution facilitating scientific interpretation is one of maximizing the number of vanishing projections. (This concept is frequently referred to as "simple structure", however this term is somewhat misleading and will not be used in this paper.) Either the number of vanishing perpendicular projections or the number of vanishing parallel projections must be maximized, inasmuch as both types of projections cannot generally be maximized within the context of a single (either primary or reference) system. That is to say, the zero "loadings" in the pattern and the structure matrix cannot both be maximized within a single system. The matrix in which the vanishing projections are to be maximized is dependent upon the interpretation that one wishes to make from the final solution. If the interpretation is to be made in terms of the correlations between the variables and the factors then the vanishing perpendicular projections of the variable vectors onto the unit reference vectors, the near-zero entries of the reference structure matrix, should be maximized. If one wishes to treat the observed variables as dependent variables and the factors as independent variables, then the vanishing parallel projections of the variable vectors onto the unit primary vectors, the near-zero entries of the primary pattern matrix, should be maximized.

In the past, with several exceptions, most attempts to develop analytic oblique procedures have followed the Thurstonian mode of thought, maximizing the vanishing perpendicular projections of the variable vectors onto the unit reference vectors. The reason for this is not particularly clear, but either the Thurstonian approach is less complex than the Holzinger approach or factor analysts find the reference structure matrix an easier matrix to interpret. In keeping with tradition this paper will follow the Thurstonian model, however it should become apparent to the reader, as a result of reading this paper, that the choice of model in this paper was somewhat arbitrary as both types of solutions may be computed with ease using the new transformation procedure developed and presented herein.

The objective of this paper is to acquaint the reader with certain aspects of the methodology, properties and nature of the general obliquimax transformation (Hofmann, 1971). For pedagogical and illustrative purposes one of Thurstone's methods of determining oblique transformations is presented and then modified to produce a simplified version of the obliquimax which is referred to as the simplified obliquimax.

The simplified obliquimax is unique in the sense that it is a semi-subjective transformation procedure that depends neither on an oblique analytic simple structure criterion nor graphical techniques to determine the oblique transformation solution. It provides a conceptually simple yet reliable oblique transformation procedure for most sets of data. However, it is *not intended* to be a practical working model. It is the general obliquimax that is the practical model.

This paper is composed of three sections. In the first section, (Section I), Thurstone's (1947) method of determining subjective oblique transformations is discussed within the context of two-dimensional sections. Although nothing in the first section is new, it defines and describes geometrically the matrices and terminology traditionally used in the Thurstonian type oblique transformations. The first section also establishes an algebraic model for determining subjective oblique transformation solutions through the use of an iterative method.

In the second section, (Section II), the general obliquimax is briefly discussed with respect to the matrix equations defining an oblique solution. Special emphasis is placed on the matrices of direction numbers and solution matrices expressed within the metric of the original factor solution. Several important similarities between the Thurstone and Holzinger solutions are noted. This section does not provide a detailed discussion of the general obliquimax inasmuch as it is included only to provide a basic theoretical rationale for the development of the simplified obliquimax in the third section of the paper.

In the third and final section, (Section III), of this paper the simplified obliquimax is presented. Thurstone's initial matrix of direction numbers is defined as a symmetric matrix *a priori* without the use of planar plots. All subsequent iterative stage solution matrices are expressed within the metric of the original factor solution and defined in terms of an orthogonal transformation of the original factor solution and exponential powers of the initial symmetric matrix of direction numbers. Conjectures are made about certain new aspects of the geometry of oblique solutions within the framework of the direction numbers of the simplified obliquimax.

In Sections I and III a set of illustrative data is used to clarify the discussion. Iterative solutions for this data set are determined by Thurstone's method in Section I and by the simplified obliquimax in Section III.

Aside from acquainting the reader with the general obliquimax, this paper should clarify and extend certain theoretical aspects of oblique solutions in general.

Section I
Thurstone's Method Of Determining Oblique Transformation Solutions By Two Dimensional Sections:

As previously mentioned Thurstone sought to define an oblique solution with respect to perpendicular projections onto the reference vectors, thus his solution matrix of interest was the reference structure matrix. Thurstone (1947) presented several methods of oblique transformation: plotting the normalized variable vectors onto a hyper-sphere, two dimensional sections and by three dimensional sections. His first method was quite subjective while his other two methods were primarily analytic and algebraically the principle involved in both methods is the same. In this section his algebraic principles will be used and discussed within the context of two-dimensional sections. (Although there are numerous modifications and rewordings this section is taken directly from Thurstone (1947, p. 194-224). Reference to Thurstone (1947) is implicit throughout this section.)

Assume some factor loading matrix, F, defining the perpendicular projections of n variable vectors onto r mutually orthogonal factor axes. The r axes are arbitrarily orthogonal axes as determined by the initial factoring method. For illustrative purposes Thurstone's technique will be discussed within the framework of his classic box problem (Thurstone, 1947, pp. 140-144). The centroid solution for the box problem is reported in Table 1. For this particular set of data (n = 20) and (r = 3).

Table 1

Centroid Solution* of Box Problem
Matrix F

	A_o	B_o	C_o
	A_1	B_1	C_1
1	.659	-.736	.138
2	.725	.180	-.656
3	.665	.537	.500
4	.869	-.209	-.443
5	.834	.182	.508
6	.836	.519	.152
7	.856	-.452	-.269
8	.848	-.426	.320
9	.861	.416	-.299
10	.880	-.341	-.354
11	.889	-.147	.436
12	.875	.485	-.093
13	.667	-.725	.109
14	.717	.246	-.619
15	.634	.501	.522
16	.936	.257	.165
17	.966	-.239	-.083
18	.625	-.720	.166
19	.702	.112	-.650
20	.664	.536	.483

*Thurstone, 1947, p. 194.

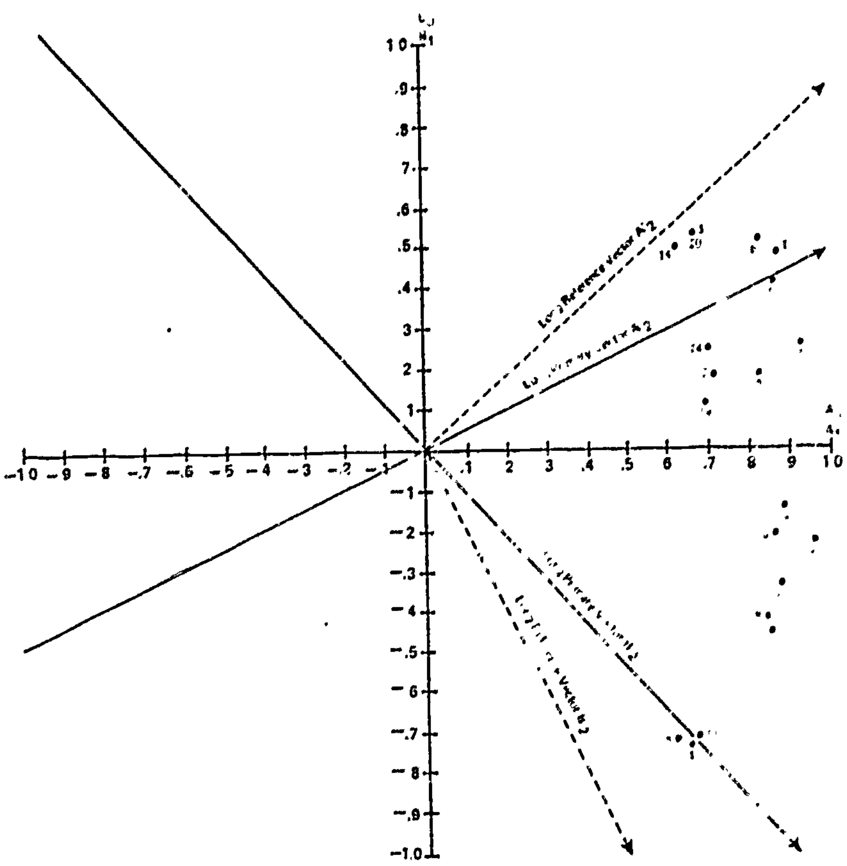
The r axes may be denoted as A_o, B_o, and C_o. These arbitrary axes are regarded as fixed in position. The problem is to select by u successive approximations the unit reference vectors, A_u, B_u, and C_u, such that the number of variable vectors with zero perpendicular projections onto these unit reference vectors is a maximum.

In starting the transformation procedure the r unit reference vectors are assumed to be mutually orthogonal and collinear with the r arbitrary orthogonal axes. That is to say, A_1 is orthogonal to B_1 and

C_1 and it is collinear with A_0. This may be observed in Figures 1, 2 and 3 which represent the planar plottings of the variable points with respect to the initial factor axes as reported in Table 1 (Figure 1 being the points as defined by the first two columns of Table 1). Note that A_0 and A_1 are the same axis.

Figure 1

Planar Plots of Variable Vector Termini
With Respect to Unit Reference Vectors
A_1 and B_1, Projected Onto Plane A_1B_1



The locations of the reference vectors may be defined with respect to the fixed orthogonal frame through the use of the matrix of direction cosines V_u. The subscript of V refers to the u-th given positions of the reference vectors (the iterative stage). When u is unity the reference vectors are collinear with the fixed orthogonal frame and V_1 is an identity matrix. The columns of V_1 give the direction cosines of the initial locations (1) of the unit reference vectors with respect to the fixed orthogonal frame.* For any matrix of direction cosines, V_u, the entry v_{ij} refers to the cosine of the angle of inclination between the unit reference vector j and the original fixed axis i.

The points in Figure 1 show the configuration of variable vector termini as they would appear when projected orthogonally onto the plane of A_1B_1. If vector A_1 is transformed in the plane A_1B_1 to the position of A_2', it will determine a plane (of hyperplane if $r > 3$) which will intersect the plane A_1B_1 in the line which is marked B_2'-primary. The vectors associated with 1, 13 and 18 will have near-zero projections (vanishing projections) on the vector A_2'. It is important to note that the B_2'-primary passes through the group of points 1, 13 and 18. Similarly the given position of B_1 can be transformed in the A_1B_1 plane to B_2' and its associated plane will intersect the plane A_1B_1 in the line marked A_2'-primary. The A_2'-primary passes through the group of points 6, 9 and 12 and their variable vectors have vanishing projections on B_2'.

In transforming A_1 and B_1 to the positions A_2' and B_2' respectively new positions have been estimated graphically for the reference vectors A and B such that the number of variable vectors having vanishing projections has increased. It is important to note here that A_1 and B_1 are in part bases and altitudes of right triangles whose hypotenuses are A_2', B_2', A_2'-primary and B_2'-primary. The vectors A_2', B_2', A_2'-primary

*The initial factor loading matrix F is assumed to represent the first iterative stage of Thurstone's solution. Technically F should be subscripted as F_1, however for convenience the subscript 1 is omitted.

and B_2'-primary are not of unit length. The prime is used to signify that A_2' and B_2' are "long reference vectors" and that A_2'-primary and B_2'-primary are "long primary vectors".

The geometric discussion in this section is not quite the same as that presented by Thurstone. It is hoped that by including the long primary vectors as well as the long reference vectors some of the geometric similarities between the Thurstone and Holzinger solutions will become evident. Technically a reference vector is orthogonal to a hyperplane of $(r - 1)$ dimensions. In any r-dimensional space there are r hyperplanes of $(r - 1)$ dimensions, and therefore r reference vectors. The intersection of $(r - 1)$ hyperplanes defines a primary vector, therefore in any r-dimensional space there are r primary vectors. The vectors drawn orthogonal to a hyperplane will necessarily be orthogonal to any vectors contained within the hyperplane, thereby implying that each primary vector must be orthogonal to $(r - 1)$ reference vectors. Therefore each primary vector is correlated with only one reference vector. Orthogonal to the one hyperplane not containing the primary vector is that reference vector. Within the context of this paper each reference vector is referred to by a subscripted Roman letter. The Roman letter may be thought of as representing the hyperplane to which the reference vector is orthogonal. Each long primary vector is referred to by a subscripted Roman letter. The Roman letter associated with the long primary vector may be thought of as representing the hyperplane which does not contain the primary vector. Thus for the illustrative example long primary A_2' is orthogonal to all long reference vectors with the exception of long reference vector A_2'. This discussion

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may be generalized to any number of dimensions* and will be presented algebraically in Sections II and III. At this point we only wish to call the reader's attention to the long primaries in the plotted figures and to note that they are the Holzinger (Holzinger and Harman, 1941) long primaries.

The coordinates of the termini of A_2', B_2', A_2'-primary and B_2'-primary can be defined with respect to the fixed orthogonal axes A_0, B_0 and C_0 or with respect to A_1, B_1 and C_1. Only the coordinates of the long reference vectors will be discussed in this section. The termini of the long reference vectors A_2' and B_2' are linear combinations of A_1 and B_1. Specifically:

$A_2' = 1.00A_1 + .90B_1$;

$B_2' = .50A_1 - 1.00B_1$.

The coordinates of the terminus of A_2' with respect to A_1 and B_1 are (1.00, .90) and the coordinates of B_2' are (.50, -1.00). The use of A_0 and A_1 may be somewhat perplexing to the factor analyst unfamiliar with Thurstone's methodology. For subsequent iterations the role of A_0 as opposed to the role of the previous position of the reference vector, A_{u-1} which is A_1 for the first iterative stage, will become much clearer.

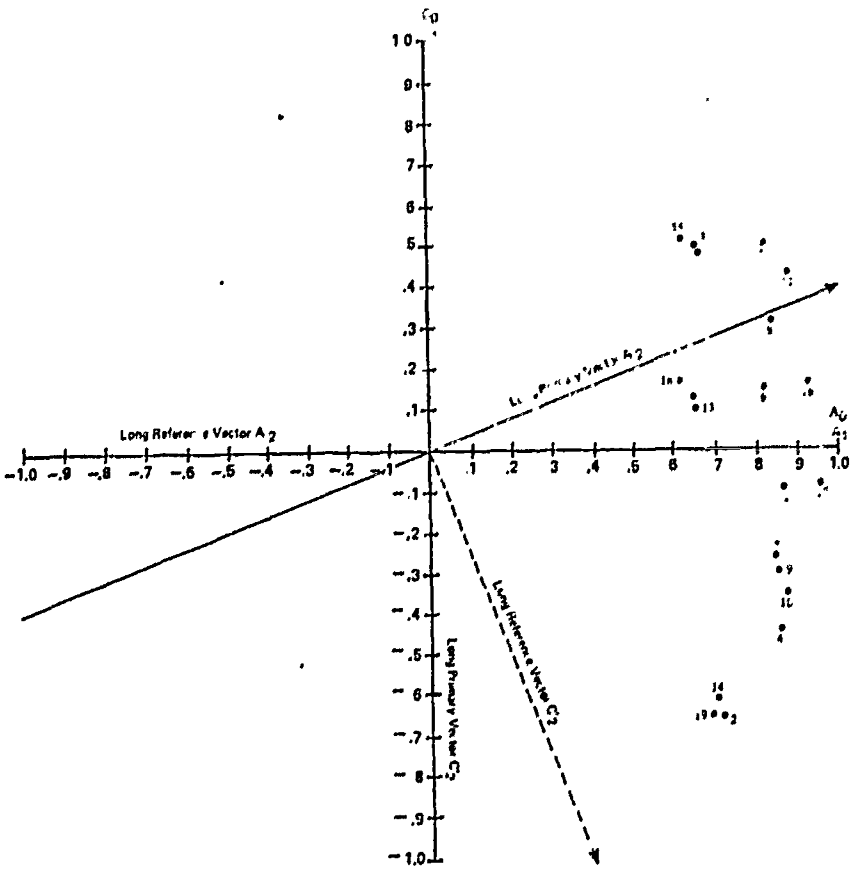
In Figure 2 the first and third columns of F have been plotted. The vector C_1 has been transformed to C_2' such that A_2'-primary passes through the group of points 8, 11 and 18. The variable vectors associated with these points will have vanishing projections on the long reference vector C_2'. The coordinates of the terminus of C_2' with respect to A_1 and C_1 are (.40, -1.00).

$C_2' = .40A_1 - 1.00C_1$

*When $r < 4$ the hyperplanes become planes.

Figure 2

Planar Plots of Variable Vector Termini
With Respect to Unit Reference Vectors
A_1 and C_1, Projected Onto Plane A_1C_1



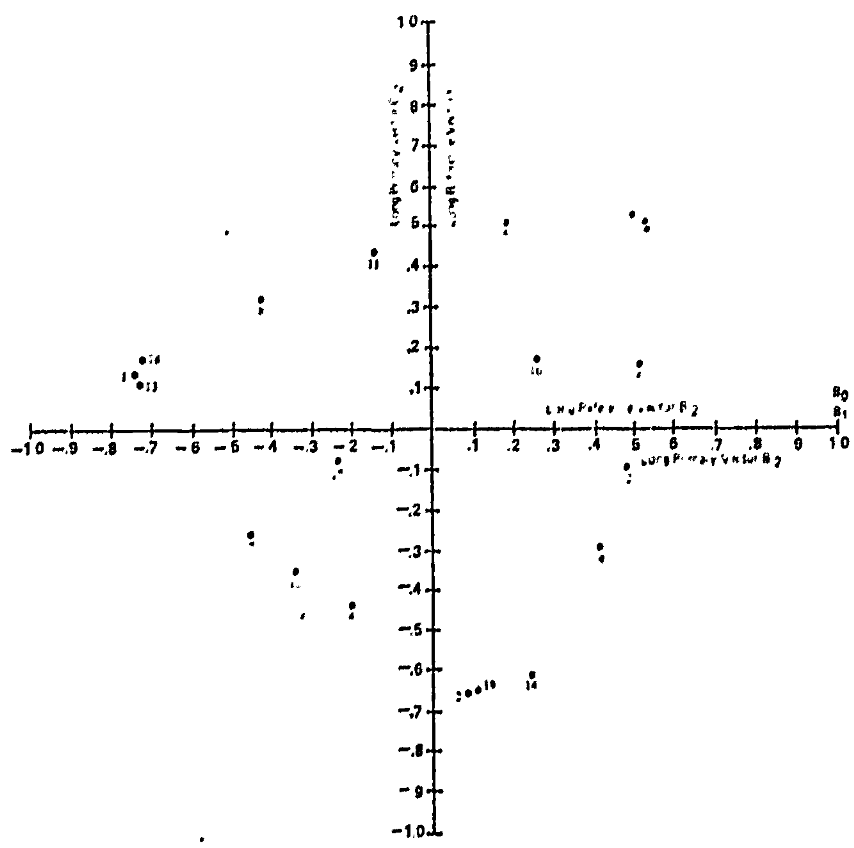
Note that A_1 is not transformed in the A_1C_1 plane. The axis A_1 in the A_1C_1 plane represents the projection of long reference vector A_2' onto the plane.

In Figure 3 the second and third columns of F have been plotted. In the B_1C_1 plane the axes A_1 and B_1 are not transformed, thus, in this plane B_1 becomes long reference vector B_2' and C_1 becomes long reference vector C_2'. The axes B_1 and C_1 represent the projections of long reference vectors B_2' and C_2', respectively, onto the B_1C_1 plane.

Although Thurstone has described the transformation process in two-dimensional sections the long reference vectors are theoretically linear combinations of all three of the initial reference vectors. They are also a linear combination of all three of the original axes.

Figure 3

Planar Plots of Variable Vector Termini
With Respect to Unit Reference Vectors
B_1 and C_1, Projected Onto Plane B_1C_1



The equations for A_2', B_2' and C_2' are correctly expressed as:

$A_2' = 1.00A_1 + .90B_1 + 0.0\ C_1$;

$B_2' = .50A_1 - 1.00B_1 + 0.0\ C_1$;

$C_2' = .40A_1 + 0.0\ B_1 - 1.00C_1$.

(Thurstone, 1947, pp. 197-198)

The coordinates of the termini of the new long reference vectors (2) with respect to the previous unit length reference vectors (1) are referred to as the direction numbers of the reference vectors with respect to the previous reference vectors. The matrix of such direction numbers will be referred to as S_{mu}, where the subscript u refers to the reference vectors of the present iterative stage and the subscript m refers to the previous iterative stage ($m = u - 1$). The matrix of such direction numbers for the second iterative stage of the illustrative example is reported in Table 2.

Table 2

Direction Numbers of Second Iterative Stage Reference Vectors With Respect To Previous Iterative Reference Vectors* - Matrix S_{12}

	A'_2	B'_2	C'_2
A_1	1.000	0.500	0.400
B_1	0.900	-1.000	0.000
C_1	0.000	0.000	-1.000

*It is assumed that F represents the first iterative stage. Thurstone, 1947, p. 198

The entries of S_{mu} by column are the coordinates of the reference vectors with respect to the reference vectors of the previous stage.

A second matrix of direction numbers is used by Thurstone. The second matrix, L_u, represents by columns the coordinates of the termini of the reference vectors of the u-th iterative stage with respect to the axes of the fixed orthogonal frame. The matrix L_u is the product of the previous matrix of direction cosines, V_m, post-multiplied by the present matrix of direction numbers, S_{mu}, of the reference vectors with respect to the m-th set of reference vectors.

$$L_u = V_m S_{mu} \quad [1]$$

For the second iterative stage V_1 is an identity matrix therefore L_2 is identical to S_{12}, however this identity will not hold for subsequent iterative stages.

As previously noted the primes on the new reference vectors distinguish the long reference vectors from unit length reference vectors. The length of the long reference vectors can be determined through an r-dimensional extension of the Pythagorean theorem: the sum of the squared lengths of the legs of a right triangle is equal to the squared length of the hypotenuse. Within the present framework the length of a long reference vector is determined with respect to the fixed orthogonal frame. Each coordinate of the terminus of a reference vector is analogous to the length of one of the legs of a right triangle whose hypotenuse is the long reference vector. The direction numbers of interest for determining the lengths of the long reference vectors are the column entries of L_u. The squared length of long reference vector A_2' is just the sum of the squared entries of the first column of L_2 (For this particular iterative stage $L_2 = S_{12}$ and the column sums of squares may be computed directly from S_{12}, Table 2.).

Let the non-zero entries of the positive definite diagonal matrix D_u^2 represent the squared lengths of the long reference vectors determined by the u-th iteration. The equation for computing D_u^2 is:

$$D_u^2 = diagonal\ (L_u'L_u). \qquad [2]$$

For the u-th iterative stage the value d_{ii} represents the length of the i-th long reference vector. For the illustrative example the diagonal entries of D_2 are reported in Table 3. Post-multiplying L_u by D_u^{-1} will rescale the metric of the direction numbers such that the column sums of squares will be unity for the matrix product $L_uD_u^{-1}$. Within a trigonometric framework the rescaling of the columns of L_u is tantamount to dividing the length of each leg of a right triangle by its hypotenuse, thereby converting the direction numbers to direction cosines.

Table 3

Lengths of the Long Reference Vectors Determined in the Second Iterative Stage* - Matrix D_2

A_2	B_2	C_2
1.345	0.000	0.000
0.000	1.118	0.000
0.000	0.000	0.929

*Thurstone, 1947, p. 198

The lengths of the long reference vectors have not been changed. The metric of the long reference vectors has been changed. In changing the metric of the coordinates of the termini of the reference vectors each coordinate becomes the cosine of the angle of inclination between a particular reference vector and a fixed axis. The rescaling of L_u by D_u^{-1} normalizes the columns of L_u to form the matrix of direction cosines V_u. The equation for computing the direction cosines associated with the unit reference vectors of the u-th iteration is:

$$V_u = L_u D_u^{-1}. \quad [3]$$

Table 4

Direction Cosines of Second Iterative Stage* - Matrix V_2

	A_2	B_2	C_2
A_o	.743	.447	.371
B_o	.669	-.894	.000
C_o	.000	.000	-.928

*Thurstone, 1947, p. 198

To determine the perpendicular projections of the n variable vectors onto the r unit length reference vectors as estimated by the u-th iterative stage it is necessary to post-multiply F by the matrix V_u. Postmultiplying F by V_u (Equation 4) will transform the initial reference vectors into the positions of the u-th estimate of the reference vectors and the resulting matrix, F_u, will be the reference structure matrix for the u-th iterative stage. For the illustrative example V_2 is reported in Table 4 and F_2 is reported in Table 5.

$$F_u = FV_m S_{mu} D_u^{-1}$$

$$F_u = FL_u D_u^{-1}$$

$$F_u = FV_u \qquad [4]$$

Table 5

Reference Structure Matrix as Determined By the Second Iterative Stage* - Matrix F_2

	A_2	B_2	C_2
1	-.003	.953	.116
2	.659	.163	.878
3	.853	-.183	-.217
4	.506	.575	.734
5	.741	.210	-.162
6	.968	-.090	.169
7	.334	.787	.567
8	.345	.760	.018
9	.918	.013	.597
10	.426	.698	.655
11	.562	.529	-.075
12	.975	-.042	.411
13	.011	.946	.146
14	.697	.101	.840
15	.806	-.164	-.249
16	.867	.189	.194
17	.558	.645	.435
18	-.017	.923	.078
19	.597	.214	.864
20	.852	-.182	-.207

*Thurstone, 1947, p. 198

Once the reference structure matrix has been determined for the u-th iterative stage Thurstone (1947, p. 205) suggested that the cosines of the angular separations of the new reference vectors be assessed. His objective in assessing these cosines was one of being sure that none of the reference vectors had coalesced. If two reference vectors have coalesced the cosine of their angular separation, their correlation, will be unity and it may be assumed that the problem of transforming to singularity has occurred.

The cosine of the angle of inclination between any two reference vectors, their correlation, is the scalar product of their paired direction cosines with respect to the fixed orthogonal frame. Equation 5 may be used to determine the matrix of intercorrelations, Y_u, of the reference vectors of the u-th iterative stage.

$$Y_u = V_u'V_u$$
$$Y_u = D_u^{-1}L_u'L_uD_u^{-1}$$
$$Y_u = D_u^{-1}S_{mu}'V_m'V_mS_{mu}D_u^{-1} \qquad [5]$$

The intercorrelations of the reference vectors as determined in the second iterative stage for the illustrative example are reported in Table 6.

Table 6

Intercorrelations of Unit Length Reference Vectors as Determined in the Second Iterative Stage* - Matrix Y_2

	A_2	B_2	C_2
A_2	1.000	-0.266	0.276
B_2	-0.266	1.000	0.166
C_2	0.276	0.166	1.000

*Thurstone, 1947, p. 198

If the magnitudes of the small reference structure values in the matrix F_u have diminished with respect to F_m and if the magnitudes of the large reference structure values in F_u have increased with respect to F_m it is reasonable to proceed to the next iterative stage. However, if two unit length reference vectors are highly correlated it is most prudent to stop at this particular stage.

Assessing F_2 with respect to F (Tables 5 and 1) and seeing that the magnitudes of the entries in Table 6 are small we may progress to the next iterative step for the illustrative example.

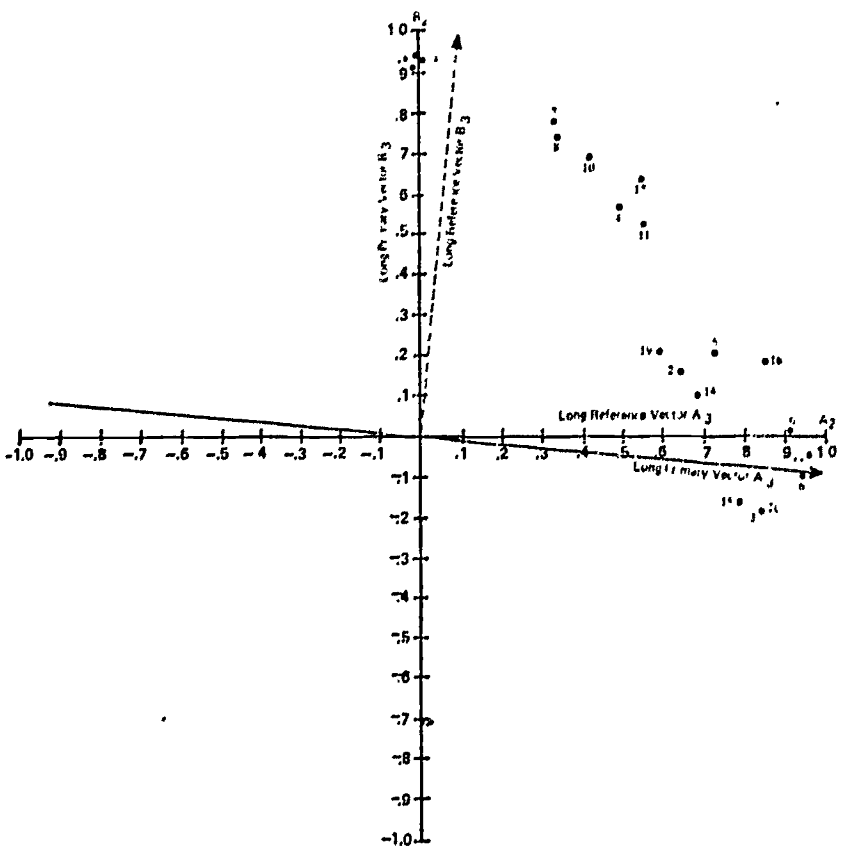
Subsequent Iterative Stages

Plot a new set of diagrams for all pairs of columns of F_u and examine them to determine the adjustments which will be made in this particular iterative stage. It is important to note here that Thurstone plotted F_u on orthogonal coordinate cross-section paper, even though the reference vectors were oblique. Thus, the reference vectors of F_u were plotted as being orthogonal. The logic behind this procedure although basically simple is frequently quite confusing. For conceptual purposes it may be assumed that the axes remain invariant and that it is the configuration of variable vectors that is being transformed. The apparent paradox is associated only with the plotting of the configuration, not with the algebra or interpretation of the reference structure loadings. (The initial papers on the obliquimax were all written within this conceptual framework and a majority of the algebra was also interpreted within this framework.) The new sets of diagrams for all pairs of columns of F_2 are presented in Figures 4, 5 and 6.

In Figure 4 the termini of the variable vectors are plotted with respect to unit reference vectors A_2 and B_2. In the third iterative

Figure 4

Planar Plots of Variable Vector Termini
With Respect to Unit Reference Vectors
A_2 and B_2, Projected Onto Plane A_2B_2



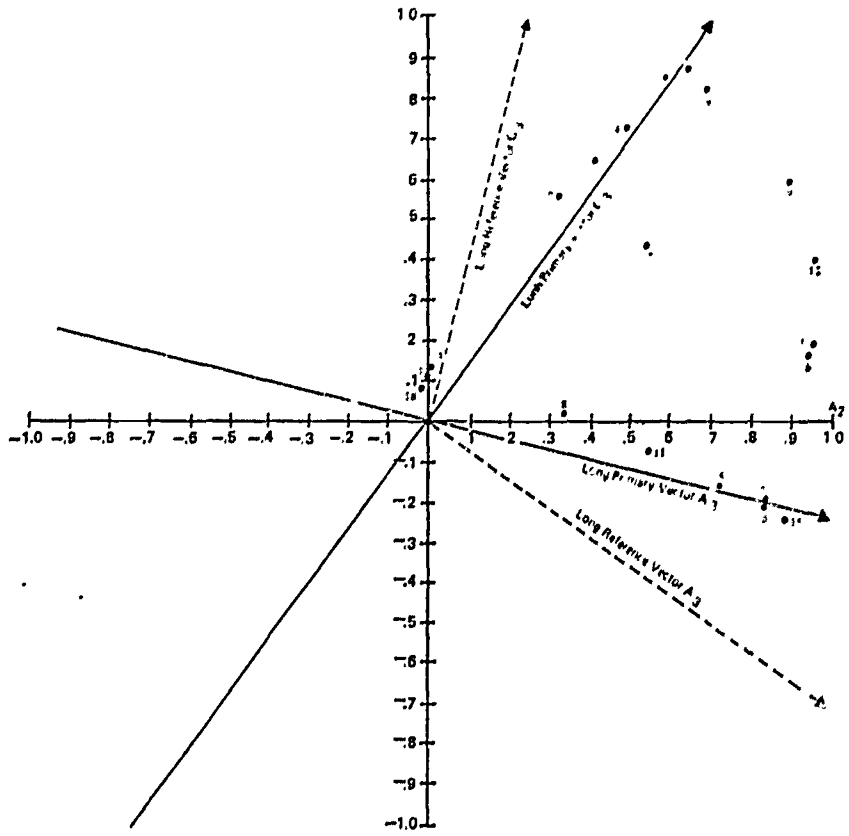
stage unit reference vector B_2 is transformed to position B_3' and A_3'-primary will pass through the group of variable points 3, 15, 20, 6, 9 and 12. The variable vectors associated with these points will have vanishing projections on long reference vector B_3'. In the A_2B_2 plane the new long reference vector B_3' may be described in terms of A_2 and B_2.

$B_3' = 1.00B_2 + .10A_2$

In Figure 5 the termini of the variable vectors have been plotted with respect to A_2 and C_2. Unit length reference vector A_2 has been transformed to A_3'. Notice that C_3'-primary still passes through the variable points 13, 1 and 18 and additionally through the points 7, 10, 4, 9, 2 and 14, thus increasing from three to nine the number of variables with vanishing

Figure 5

Planar Plots of Variable Vector Termini
With Respect to Unit Reference Vectors
A_2 and C_2, Projected Onto Plane A_2C_2



projections on the long reference vector A_3'. The new long reference vector A_3' in the A_2C_2 plane may be described in terms of the unit reference vectors A_2 and C_2.

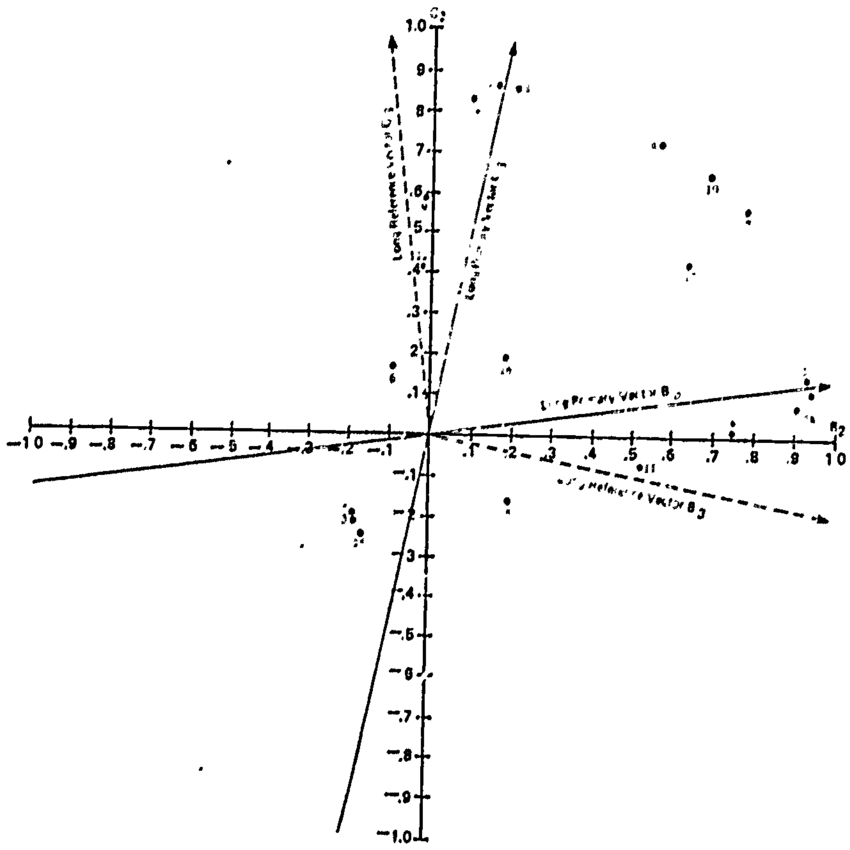
$A_3' = 1.00A_2 - .72C_2$

In the same figure C_2 has been transformed to C_3' such that the A_3'-primary passes through the group of variable points 3, 5, 15 and 20. The variables 13, 1 and 18 retain their vanishing projections on C_3' and additionally variables 3, 5, 15 and 20 also have vanishing projections on the long reference vector C_3'. The number of vanishing projections on the C reference vector has increased from three to seven in the A_2C_2 plane. Long reference vector C_3' may be described in terms of the unit reference vectors A_2 and C_2.

$C_3' = 1.00C_2 + .25A_2$

Figure 6

Planar Plots of Variable Vector Termini
With Respect to Unit Reference Vectors
B_2 and C_2, Projected Onto Plane B_2C_2



In Figure 6 the termini of the variable vectors have been plotted with respect to unit reference vectors B_2 and C_2. Unit reference vector C_2 has been transformed to C_3' so that the B_3'-primary passes through the variable points 1, 13 and 18. The new long reference vector C_3' in the B_2C_2 plane may be described with respect to unit reference vectors B_2. and C_2.

$C_3' = 1.00C_2 - .12B_2$

In the same figure unit reference vector B_2 is transformed to B_3' such that the C_3'-primary passes through the group of variable points 2, 14 and 19. The variable vectors 2, 14 and 19 will have vanishing projections on the long reference vector B_3'. The new long reference vector B_3' in the C_2B_2 plane may be described with respect to unit reference vectors B_2 and C_2.

$B_3' = 1.00B_2 - .20C_2$

It is not at all uncommon for a reference vector to be transformed in numerous planes during a single iterative stage if ($r > 3$), inasmuch as there are $r(r - 1)/2$ possible planar transformations. In this iterative stage two of the reference vectors were transformed in two planes, thus they must be described with respect to all three unit reference vectors. The final descriptions of the new long reference vectors with respect to the previous unit length reference vectors A_2, B_2 and C_2 are:

$A_3' = 1.00A_2 + 0.00B_2 - .72C_2$;
$B_3' = .10A_2 + 1.00B_2 - .20C_2$;
$C_3' = .25A_2 - .12B_2 + 1.00C_2$.
(Thurstone, 1947, p. 209)

Thurstone (1947, p. 208) referred to the coefficients of the above equations as "corrections". The implication was that the new transformations of the reference vectors are simply corrections of the previous transformation. It is prudent to realize that these corrections as reported in this paper were determined subjectively by Thurstone. He has numerous subjective "rules of thumb" that he utilized when determining these corrections (See Thurstone, 1947, pp. 207-210; 212-216).

The coefficients of the three linear equations are the direction numbers for the three new long reference vectors with respect to the previous unit length reference vectors. That is to say, the coefficients are the entries of the matrix of direction numbers S_{23}, $m = 2$ and $u = 3$.

The entries of Table 7 by column represent the coordinates of the termini of the new long reference vectors with respect to the previous

Table 7

Direction Numbers of Third Iterative Stage Reference Vectors with Respect to the Unit Reference Vectors of the Second Iterative Stage* - Matrix S_{23}

	A_3'	B_3'	C_3'
A_2	1.000	0.100	0.250
B_2	0.000	1.000	-0.120
C_2	-0.720	-0.200	1.000

*Thurstone, 1947, p. 209

unit reference vectors. The coordinates of the terminus of B_3' with respect to A_2, B_2 and C_2 would be (.10, 1.00, -.20).

To convert the direction numbers S_{23} to the matrix of direction numbers with respect to the fixed orthogonal frame, L_3, Equation 1 is used.

$$L_3 = V_2 S_{23}$$

In Table 8 the direction numbers of the new long reference vectors A_3', B_3' and C_3', are reported with respect to the axes of the fixed orthogonal framework, A_o, B_o and C_o.

Table 8

Direction Numbers Of Third Iterative Stage Reference Vectors With Respect To The Original Fixed Orthogonal Framework* - Matrix L_3

	A_3'	B_3'	C_3'
A_o	.476	.447	.503
B_o	.669	-.829	.275
C_o	.668	.186	-.928

*Thurstone, 1947, p. 209

The entries of L_3, Table 8, by column represent the coordinates of the termini of the long reference vectors with respect to the axes 1_o, B_o and C_o. With respect to the original axes A_o, B_o and C_o the coordinates for the terminus of C'_3 are (.503, .275, -.928).

Table 9

Lengths Of The Long Reference Vectors Determined In The Third Iterative Stage* - Matrix D_3

A'_3	B'_3	C'_3
1.06	.00	.00
.00	.96	.00
.00	.00	1.09

As in the previous iterative stage it is necessary to rescale the metric of the long reference vectors to that of unit length reference vectors. The squared lengths of the new long reference vectors, D^2_3, are computed through the use of Equation 2, Table 9. Matrix L_3 is column normalized to form the matrix of direction cosines, V_3, of the new unit length reference vectors with respect to the fixed orthogonal frame, Equation 3. The matrix of direction cosines for the third iterative stage is reported in Table 10.

Table 10

Direction Cosines of Third Iterative Stage - Matrix V_3

	A_3	B_3	C_3
A_o	0.450	0.466	0.461
B_o	0.632	-0.864	0.252
C_o	0.631	0.194	-0.851

*Thurstone, 1947, p. 209

The reference structure matrix associated with the third iterative stage, F_3, is computed using Equation 4. The reference structure matrix F_3 is reported in Table 11.

Table 11

Reference Structure Matrix As Determined By The Third Iterative Stage* - Matrix F_3

	A_3	B_3	C_3
1	-.082	.970	.001
2	.026	.055	.938
3	.954	-.058	.016
4	-.021	.500	.723
5	.811	.330	-.002
6	.800	-.031	.387
7	-.071	.737	.510
8	.314	.825	.011
9	.462	-.017	.756
10	-.043	.636	.621
11	.582	.626	.002
12	.642	-.030	.605
13	-.090	.958	.032
14	.088	.001	.919
15	.931	-.037	-.026
16	.688	.246	.356
17	.231	.641	.456
18	-.070	.946	-.035
19	-.024	.104	.905
20	.945	-.060	.026

*Thurstone, 1947, p. 209

The intercorrelations of the new unit length reference vectors are computed using Equation 5. The matrix of intercorrelations of the unit length reference vectors, Y_3, as determined in the third iterative stage is reported in Table 12.

The magnitudes of the loadings in Table 11 are assessed with respect to the loadings in Table 5. Clearly the small loadings are diminishing and the large loadings are increasing. None of the off-diagonal entries

Table 12

Intercorrelations of Unit Length Reference Vectors
As Determined By The Third Iterative Stage* - Matrix Y_3

	A_3	B_3	C_3
A_3	1.000	-0.214	-0.171
B_3	-0.214	1.000	-0.169
C_3	-0.171	-0.169	1.000

*Thurstone, 1947, p. 209

of Y_3 are approaching unity. A fourth iterative stage is in order.

A detailed discussion of the fourth iterative stage will not be presented in this paper. As with the previous iterative stages the columns of the previously determined reference structure matrix are plotted. The direction numbers of the new long reference vectors are estimated. The direction cosines are determined and the fourth iterative stage reference structure matrix is computed from F. The fourth iterative stage is the final iterative stage for the illustrative example. The reference structure matrix and the reference vector intercorrelation matrix as determined by the fourth iterative are presented in Tables 13 and 14 respectively.

Table 13

Reference Structure Matrix As Determined
By The Fourth Iterative Stage* - Matrix F_4

	A_4	B_4	C_4
1	-.006	.965	.001
2	.032	.009	.938
3	.964	-.010	.016
4	.025	.462	.723
5	.854	.370	-.002
6	.810	-.009	.387
7	-.003	.707	.510
8	.395	.840	.011
9	.468	-.031	.756
10	.015	.602	.621
11	.649	.654	.002
12	.649	-.027	.605
13	-.003	.951	.032
14	.090	-.040	.919
15	.943	.012	-.026
16	.721	.263	.356
17	.294	.629	.456
18	.016	.943	-.035
19	-.014	.058	.905
20	.955	-.013	.026

*Thurstone, 1947, p. 213

Table 14

Intercorrelations of Unit Length Reference Vectors
As Determined In The Fourth Iterative Stage* - Matrix Y_4

	A_4	B_4	C_4
A_4	1.000	-0.066	-0.188
B_4	-0.066	1.000	-0.226
C_4	-0.188	-0.226	1.000

*Thurstone, 1947, p. 213

Additional Algebraic Aspects of Thurstone's Iterative Methodology:

After the first iterative stage Thurstone defined, algebraically, a second type of transformation matrix H_{mu}. This transformation matrix, H_{mu}, was used in conjunction with F_m, the previously computed reference structure matrix, to compute the u-th reference structure matrix, F_u. The matrix equations for computing F_u, disregarding H_{mu}, are reported by Equation 4.

$$F_u = FV_m S_{mu} D_u^{-1}$$

$$F_u = FL_u D_u^{-1}$$

$$F_u = FV_u \qquad [4]$$

Thurstone defined L_u in terms of V_m and S_{mu}, equation 1. The matrix L_u is geometrically meaningful. However, H_{mu} is defined in terms of S_{mu} and D_u^{-1} and does not appear to be geometrically meaningful.

$$H_{mu} = S_{mu} D_u^{-1} \qquad [5]$$

The matrix F_u is defined as the product of the previous reference structure matrix, F_m, post-multiplied by H_{mu}.

$$F_u = F_m H_{mu} \qquad [6]$$

$$F_u = FV_m H_{mu} \qquad [7]$$

The matrix H_{mu} therefore transforms F_m to F_u. The elements of H_{mu} are neither direction cosines nor direction numbers. The entries of S_{mu} are expressed within the metric of the m-th stage unit reference vectors while the entries of D_u^2 are expressed within the metric of the original fixed orthogonal frame. Any transformation matrix may be expressed as a product of all previous H-matrices (Thurstone, 1947, p. 206).

$$V_u = (H_{01})(H_{12})(H_{23})(H_{34})\ldots(H_{mu}) \qquad [8]$$

For the illustrative example the H-matrices associated with third and fourth iterative stages are reported in Tables 15 and 16. For the second iterative stage H_{12} is identical to V_3.

Table 15

H-Matrix Computed For The Third Iterative Stage* - Matrix H_{23}

0.945	0.104	0.229
0.000	1.042	0.110
-0.680	-0.208	0.917

*Thurstone, 1947, p. 198

Table 16

H-Matrix Computed For The Fourth Iterative Stage* - Matrix H_{34}

1.016	.050	.000
.091	.999	.000
.000	-.050	1.000

*Thurstone, 1947, p. 209

The function of H_{mu} may be thought of as providing an alternative method of computing F_u directly from F_m as opposed to computing F_u from F through the use of V_u.

Summary of Section 1:

Thurstone's (1947) method of determining oblique transformations has been presented within the context of two dimensional sections. Through the use of an illustrative problem his terminology and matrices were discussed. Although certain aspects of Thurstone's approach were modified it may be assumed that the discussion presented in this section was taken from Thurstone (1947, pp. 194-224) and typifies his approach to oblique transformation solutions.

There are numerous objections to Thurstone's procedures, all of which are either directly or indirectly associated with the matrix S_{mu}. The matrix S_{mu} is the subjective matrix of direction numbers defining the termini of the u-th iterative stage long reference vectors with respect to the unit length reference vectors of the m-th iterative stage. It would be extremely difficult for two factor analysts working independently on the same factor matrix to determine identical oblique solutions when using Thurstone's technique. It would be a most arduous task for a beginning student to apply Thurstone's methodology successfully to a set of data in which $(r > 3)$. Thurstone's method becomes prohibitive timewise as the number of factors increases inasmuch as $r(r - 1)/2$ plots are required at any one iterative stage to determine S_{mu}. It is conceivable that a bad estimate of S_{mu} might be obtained at some early iterative stage and not be recognized as such for several iterative stages. Finally, Thurstone's method is just too time consuming and unreliable for all except the most experienced factor analyst.

Section II
A Basic Theoretical Rationale For The Simplified Obliquimax As Provided By The General Obliquimax

In this section the general obliquimax is briefly discussed with respect to the matrix equations defining an oblique solution. Special emphasis is placed on the matrix of direction numbers, L_u as discussed in Section I, and the solution matrices expressed within the metric of the original factor solution. This section discusses oblique solutions within the framework of variance modification and allocation. The function of this section is to provide a basic theoretical rationale for the development of the simplified obliquimax in Section III.

Common Variance Modification and Allocation:

In determining an oblique transformation, a majority of the reference vectors, if not all of them, will covary having non-zero perpendicular projections upon each other as opposed to an orthogonal transformation in which none of the factor axes covary. Within the obliquimax framework the total common variance associated with an initial factor solution is defined as the total sum of the squared projections of the variable vectors onto the initial factor axes. The common variance associated with any one of the initial common factors is defined as the sum of the squared perpendicular projections of the variable vectors onto that factor axis. An orthogonal transformation will not change the total common variance but it will in general alter the sum of the squared perpendicular projections associated with each common factor.

When working within an oblique framework the process of column normalizing the direction numbers to form the matrix of direction cosines is actually a process of converting the metric from that of the initial fixed frame to the metric of the unit length reference vectors of the particular iterative stage associated with the direction cosines. Thus, for each iterative stage of an oblique solution the metric is changed and is not comparable to the metric of the initial fixed frame. Because of this metric variation the sum of the squared perpendicular projections of the variable vectors onto the reference vectors are neither comparable between iterative stages nor comparable with the sum of the squared projections onto the factor axes of the initial fixed frame.

The metric of the initial fixed frame may be retained in an oblique solution if the matrix of direction numbers, previously defined by Equation 1, is used in place of the matrix of direction cosines, previously defined by Equation 3. The entries of the resulting "structure"* matrix would represent the perpendicular projections of the variable vectors onto the reference vectors, but the entries would be expressed within the metric of the initial fixed frame. Within Thurstone's terminology such a matrix would be referred to as a long reference vector structure* matrix inasmuch as the entries are analogous to perpendicular projections of the variable vectors onto the long reference vectors as opposed to unit length reference vectors. When the metric is held constant in this fashion, it will be observed that the total sum of the squared projections of the variable vectors onto the long reference vectors is less than the total sum of the squared projections of the variable vectors onto the initial fixed axes. In the next subsection it will be demonstrated that the difference between the sum of the squared projections of the two matrices is accounted for by the perpendicular projections of the primary vectors onto each other, the covarying of the primary vectors.

Within this framework the role of the direction numbers of the long reference vectors with respect to the initial fixed axes, L_u, is one of defining the long reference vectors in such a manner that the sum of the squared perpendicular projections of the variable vectors onto these long

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* In a latter portion of the next subsection it will be demonstrated that the long reference vector structure matrix is not the only meaningful name that might be given to the matrix being discussed.

reference vectors is less than the sum of the squares of their projections onto the initial factor axes. There should be some systematic relationship between the matrices of direction numbers determined in successive iterative stages such that the sum of the squared perpendicular projections of the variable vectors onto the long reference vectors becomes successively smaller at each stage. It will be demonstrated that such a framework will provide numerous conceptual and algebraic advantages.

Theoretical Equations for the General Obliquimax Transformation:

The objective of this subsection is to provide a basis for the computational aspects of the simplified obliquimax through a brief presentation of the equations and logic for the general obliquimax (Hofmann, 1971).

Assume some initial factor loading matrix F, defining the perpendicular projections of n variable vectors onto r mutually orthogonal factor axes as determined by the initial factoring method. The problem is to select by u, successive approximations the unit reference vectors A_u, B_u and C_u such that the number of variable vectors with vanishing projections onto these unit reference vectors is a maximum. (Where deemed necessary for illustrative purposes r will be assumed to be three.)

In the obliquimax transformation all matrices of direction numbers are defined as the product of some positive definite diagonal matrix and some orthonormal transformation matrix, T. A matrix of direction numbers developed in this manner, and the ensuing matrix of direction cosines, will always be non-singular and generally non-orthogonal. This somewhat unusual approach was first suggested by Harris and Kaiser (1964) in their classic paper on determining oblique transformation solutions through the use of orthogonal transformation matrices. Although the discussion in this paper will only encompass their case I and case II solutions, the general obliquimax encompasses all three of the cases discussed by Harris and Kaiser.

Let the ii-th element of the positive definite diagonal matrix D^2, represent the sum of the squared projections of the n variable vectors onto the i-th factor axis associated with F.

$$D^2 = diagonal\ [F'F] \qquad [9]$$

All matrices of direction numbers within the general obliquimax framework are defined specifically as the product of some exponential power, p, of D and an orthonormal matrix T. Equation 10 represents the matrix of direction numbers, L_u, for defining the termini of the u-th iterative stage long reference vectors with respect to the initial fixed frame.

$$L_u = D^{-p}T \qquad [10]$$

Along with the algebraic advantages of such a definition of L_u, there are several conceptual and interpretative advantages. In defining the positive definite diagonal matrix as some exponential function of the column sums of squares of F, the matrix of direction numbers is implicitly some function of the common variance and hence a function of the perpendicular projections of the variable vectors associated with F.

The matrix F may be rewritten as:

$$F = ED. \qquad [11]$$

The matrix E is the column normalized form of F and the elements of D are the square roots of the elements of D^2. The matrix F may be post multiplied by L_u to form F_u^* which would be the matrix whose elements represent the perpendicular projections of the variable vectors onto the long reference vectors determined by the u-th iterative stage.

$$F_u^* = FL_u = FD^{-p}T \qquad [12]$$

There are several important aspects of Equation 12 that should be noted. Any matrix of direction numbers within the general obliquimax framework is simply a row rescaling of T by some exponential power, p, of the matrix D. Equation 12 may be rewritten as Equation 13.

$$F_u^* = EDD^{-p}T \qquad [13]$$

The discussion of common variance associated with F_u^* will be based upon the sum of the squared perpendicular projections onto the r long reference vectors associated with F_u^*. Inasmuch as the columns of E are normalized, the matrix product $(DD^{-p}T)$ defines the allocation of the perpendicular projections of the variable vectors onto the long reference vectors. That is to say, the common variance, within the metric of the initial fixed frame, associated with F_u^* can be discussed with respect to the matrix $(DD^{-p}T)$.

The matrix T is an orthonormal matrix of direction cosines. Therefore, the sum of the squares of any column or row of T is unity $(T'T = TT' = I)$. The square of any cosine is a proportion. The square of any diagonal element of (DD^{-p}) is a portion of the total common variance associated with F. The variance aspects of F_u^* can be discussed through a consideration of the squared elements of $(DD^{-p}T)$. The variance discussed within this specific framework would be with respect to the sums of the squared perpendicular projections of the n variable vectors onto the r long reference vectors. A symbolic representation of the squared elements of $(DD^{-p}T)$ is reported in Table 17.

The variance contributed by fixed axis A_o to the sum of the squared perpendicular projections of the n variable vectors onto all r long reference vectors is (d_{11}^2/d_{11}^{2p}); the variance contributed by fixed axis B_o is (d_{22}^2/d_{22}^{2p}); the variance contributed by fixed axis C_o is (d_{33}^2/d_{33}^{2p}).

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Table 17

Symbolic Representation of the Squared Elements of $(DD^{-p}T)$ (assume $r = 3$)

	A'_u	B'_u	C'_u
A_o	$(d^2_{11}/d^{2p}_{11})\cos^2\beta_{11}$	$(d^2_{11}/d^{2p}_{11})\cos^2\beta_{12}$	$(d^2_{11}/d^{2p}_{11})\cos^2\beta_{13}$
B_o	$(d^2_{22}/d^{2p}_{22})\cos^2\beta_{21}$	$(d^2_{22}/d^{2p}_{22})\cos^2\beta_{22}$	$(d^2_{22}/d^{2p}_{22})\cos^2\beta_{23}$
C_o	$(d^2_{33}/d^{2p}_{33})\cos^2\beta_{31}$	$(d^2_{33}/d^{2p}_{33})\cos^2\beta_{32}$	$(d^2_{33}/d^{2p}_{33})\cos^2\beta_{33}$

(Other than noting that the exponent p will be chosen such that the total column sums of squares of F^*_u is less than the total associated with F discussion of p will be deferred at this point.) The proportion of that variance contributed by A_o, (d^2_{11}/d^{2p}_{11}), that will be allocated specifically to the sum of the squared projections of the variable vectors onto long reference vector: A'_u is $\cos^2\beta_{11}$; B'_u is $\cos^2\beta_{12}$; C'_u is $\cos^2\beta_{13}$. A similar interpretation may be made for the other $(r - 1)$ fixed axes with respect to the sum of the squared projections of the n variable vectors onto the r long reference vectors.

A portion of the total variance associated with a fixed axis is allocated to the sum of the squared perpendicular projections of the variable vectors onto the long reference vectors. The discussion here concerns the proportionate distribution of this variance by a fixed axis to each of the long reference vectors. The portion of variance associated with the j-th fixed axis that is allocated to the sum of the squared perpendicular projections of the n variable vectors onto the r long reference vectors is always (d^2_{jj}/d^{2p}_{jj}). Of that particular portion of variance, the proportionate allocation to the i-th long reference vector is $\cos^2 \beta_{ji}$, ($i = 1, 2, \ldots r$). Therefore, at any particular iterative stage the only value that will be altered in Table 17 is the exponent p. (In the definitive paper on the general obliquimax, Table 17 is further modified to provide additional information about the reference vectors.)

To further clarify the role of variance modification in the general obliquimax, it is necessary to discuss the variance covariance matrices associated with both the reference vectors and the primary vectors. Let the symmetric matrix R^*, of order n by n and singular of rank r, be the major product of F.

$$R^* = FF' \qquad [13]$$

The matrix R^* is a reproduced matrix whose off-diagonal elements approximate either the observed correlations or covariances between the n variables.

For any oblique solution Equation 14 must hold.

$$R^* = F_u(Y_u)^{-1}F'_u \qquad [14]$$

The matrices F_u and Y_u were previously defined by Equations ~~14~~ 4 and 5.

Let the matrix Z_u represent the matrix of intercorrelations associated with the r unit length primary vectors. Then by definition, Z_u is the normalized inverse of Y_u (Thurstone, 1947, p. 215). Let the matrix D_3^2 be defined as the diagonal of Y_u^{-1}, therefore pre- and post-multiplication of Y_u^{-1} by D_3^{-1} will result in the matrix of intercorrelations for the primaries.

$$D_3^2 = diagonal\ [Y_u^{-1}] \qquad [15]$$

$$Z_u = D_3^{-1}(Y_u^{-1})D_3^{-1} \qquad [16]$$

Through substitution from Equations 2, 3, 5 and 10, the matrix Y_u may be defined algebraically within the obliquimax framework.

$$Y_u = D_u^{-1}T'D^{-2p}TD_u^{-1} \qquad [17]$$

Following from Equation 17 is the algebraic definition of Z_u within the obliquimax framework.

$$Z_u = D_3^{-1}(D_u T' D^{2p} T D_u) D_3^{-1} = (D_3^{-1} D_u)(T' D^{2p} T)(D_u D_3^{-1}) \qquad [18]$$

In Equation 18, the matrix product of $(D_3^{-1} D_u)$ is just the diagonal matrix that normalizes the columns of $(D^p T)$. The matrix $(Z_u' Z_u)^{-1}$ is the variance covariance matrix, Z_u^*, associated with the long primary vectors. Thus, Z_u^* is just the inverse of Y_u^*.

$$Y_u^* = T' D^{-2p} T \qquad [19]$$

$$Z_u^* = T' D^{2p} T = (Y_u^*)^{-1} \qquad [20]$$

It is immediately apparent from Equations 19 and 20 that the only algebriac difference between the primary variance covariance matrix and the reference variance covariance matrix is the sign of the exponent p. Furthermore, the only algebraic difference between the matrix of direction numbers, $(D^p T)$, for the primary structure matrix, and the matrix of direction numbers, $(D^{-p} T)$, for the reference structure matrix, is the sign of the exponent p. Thus, the matrix of direction numbers for the long primary structure matrix is just the transpose of the inverse of the matrix of direction numbers for the long reference structure matrix. Equations 19 and 20 incorporate the metric of the initial fixed axes. Utilizing Equations 19 and 12 it is possible to redefine R^* using matrices expressed in the metric of the initial fixed axes.

$$R^* = F_u^* (Y_u^*)^{-1} F_u^{*\prime} \qquad [21]$$

By substituting Z_u^* for $(Y_u^*)^{-1}$ Equation 22 results.

$$R^* = F_u^* Z_u^* F_u^{*\prime} \qquad [22]$$

The trace of R^* represents the total common variance and it is equal to the total column sums of squares for the matrix F. Equation 22 is presented to demonstrate that the total variance associated with F may be thought of as being distributed between F_u^* and Z_u^*. In explaining Table 17 the variance associated with F_u^* was discussed with respect to the sum of the squared perpendicular projections of the n variable vectors onto the r long reference vectors. It was pointed out that the variance associated with F_u^* would always be less than the variance associated with F. It may therefore be concluded that the common variance not allocated to F_u^* must be allocated to Z_u^*. The variance allocated to Z_u^* accounts for the perpendicular projections of the r long primary vectors onto each other. The variance associated with Z_u^* can be discussed with respect to (D^pT), the matrix of direction numbers for computing the primary structure matrix. Although interest here does not center on the primary structure matrix, it may be inferred from Equation 22 that the matrix of direction numbers associated with such a matrix is instrumental in explaining variance allocation. The variance aspects of Z_u^* can be discussed through a consideration of the squared elements of (D^pT). A symbolic representation of these squared elements is reported in Table 18.

Table 18

Symbolic Representation of the Squared Elements of (D^pT) (assume r = 3)

	A_u'-Primary	B_u'-Primary	C_u'-Primary
A_o	$d_{11}^{2p} \cos^2 \beta_{11}$	$d_{11}^{2p} \cos^2 \beta_{12}$	$d_{11}^{2p} \cos^2 \beta_{13}$
B_o	$d_{22}^{2p} \cos^2 \beta_{21}$	$d_{22}^{2p} \cos^2 \beta_{22}$	$d_{22}^{2p} \cos^2 \beta_{23}$
C_o	$d_{33}^{2p} \cos^2 \beta_{31}$	$d_{33}^{2p} \cos^2 \beta_{32}$	$d_{33}^{2p} \cos^2 \beta_{33}$

The variance contributed by fixed axis A_o to the sum of the squared perpendicular projections of the long primaries onto each other is (d_{11}^{2p}); the variance contributed by fixed axis B_o is (d_{22}^{2p}); the variance contributed by fixed axis C_o is (d_{33}^{2p}). The proportion of that variance contributed by A_o, (d_{11}^{2p}), that will be allocated to the sum of the squared perpendicular projections of the $(r - 1)$ long primary vectors onto long primary vector: A_u' is $\cos^2\beta_{11}$; B_u' is $\cos^2\beta_{12}$; C_u' is $\cos^2\beta_{13}$. A similar interpretation may be made for the other $(r - 1)$ fixed axes with respect to the sum of the squared perpendicular projections of the r long primary vectors onto each other.* .

A portion of the total variance associated with a fixed axis is allocated to the sum of the squared perpendicular projections of the long primary vectors onto each other. The discussion with respect to Table 18 concerns the proportionate distribution of this variance by a fixed axis to each of the long primary vectors. The portion of variance associated with the j-th fixed axis that is allocated to the sum of the squared perpendicular projections of the r long primary vectors onto each other is always (d_{jj}^{2p}). Of that particular portion of variance, the proportionate allocation to the i-th long primary vector is $\cos^2\beta_{ji}$, $(i = 1, 2, \ldots r)$. Therefore, at any particular iterative stage, the only value that will be altered in Table 18 is the exponent p.

Comparing Tables 17 and 18, several generalizations may be made. The variance associated with the i-th fixed axis, d_{ii}^2, may be divided into two multiplicative portions (d_{ii}^2/d_{ii}^{2p}) and (d_{ii}^{2p}). One portion of the variance, (d_{ii}^2/d_{ii}^{2p}), is associated with the sum of the squared perpendicular projections of the n variable vectors onto the r long reference vectors. The second portion of the variance, (d_{ii}^{2p}), is associated with the sum of the squared perpendicular projections of the r long primary vectors onto each other. The proportionate contribution of (d_{ii}^2/d_{ii}^{2p}) made to the sum of the perpendicular projections of the n

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*The element $d_{ii}^p \cos\beta_{ij}$ also represents the perpendicular projection of the long reference vector j onto the i-th fixed axis.

variable vectors onto the j-th long reference vector is $cos^2\beta_{ij}$. The proportionate contribution of (d^{2p}_{ii}) made to the sum of the squared perpendicular projections of the $(r - 1)$ long primary vectors onto the j-th long primary vector is $cos^2\beta_{ij}$. Equation 23 defines the division of variance for the i-th fixed axis.

$$d^2_{ii} = (d^2_{ii}/d^{2p}_{ii})(d^{2p}_{ii}). \qquad [23]$$

Several observations with respect to p can be made from Equation 23. If $(p = 0)$, there will be no allocation of variance to the projections of the primaries onto each other and the associated solution will be an orthogonal transformation solution. If $(p = 1)$, all of the variance will be allocated to the projections of the primaries onto each other and the resulting oblique solution would be analogous to the Harris and Kaiser (1964) independent cluster solution. As the value of p progresses from zero to unity, the amount of variance allocated to the projections of the primaries onto each other becomes progressively larger. Thus, as p progresses from zero to unity, one should expect the primary vectors to become progressively more correlated. One might be inclined to limit the values of p to values within the interval bounded by zero and unity, but there is at this time no rationale for such a limitation. (For some sets of empirical data the general obliquimax has iterated to a value of p that is slightly larger than unity.) It is prudent to realize that for a value of p considerably larger than unity, but not necessarily larger than three, some of the diagonal entries of D^{-p} might approach zero and the matrix D^{-p} will not for all practical purposes be a positive definite matrix. (Theoretically, D^{-p} will always be positive definite however computationally some near zero values will function as zeros.)

Within the framework defined, the only algebraic value that changes from one iterative stage to the next in defining F_u^* is the exponent p. The discussion, however, has centered around the long reference vectors and long primary vectors. Traditionally an oblique solution is interpreted within the framework of either unit length reference vectors or unit length primary vectors. In order to establish the oblique solution within a traditional framework it will be necessary to compute the diagonal matrices that will rescale F_u^*, Y_u^* and Z_u^* to the metric of unit length vectors.

From Equations 2, 3 and 4 it may be inferred that the diagonal matrix for rescaling F_u^* to the unit length reference structure matrix is determined from the diagonal of $(L_u'L_u)$. In the previously mentioned equations this diagonal matrix was referred to algebraically as D_u^{-1}. Within the obliquimax framework this diagonal matrix will be referred to as D_{u1}^{-1}. The matrix D_{u1}^2 is defined by Equation 24.

$$D_{u1}^2 = diagonal\ [T'D^{-2p}T] \qquad [24]$$

Equation 4 may be rewritten as Equation 25 to define the unit length reference structure matrix within the algebraic framework of the obliquimax.

$$F_u = FD^{-p}TD_{u1}^{-1} \qquad [25]$$

In Equation 20 the variance covariance matrix of the long primary vectors, Z_u^*, was defined as the inverse of the long reference vector variance covariance matrix. It was inferred from Equations 19 and 20 that the matrix of direction numbers for transforming F to the long primary structure matrix was just the transpose of the inverse of the matrix of direction numbers necessary to transform F to the long reference structure matrix. However, the primary structure matrix is not generally used for interpretative purposes. It is the primary pattern matrix that is interpreted within Holzinger's framework. The matrix necessary to transform the initial matrix F to the primary pattern matrix, denoted as W_u, is by definition the transpose of the inverse of the primary structure transformation matrix. The primary structure transformation matrix is just the column normalized form of the direction numbers associated with the long primary vectors. Let D_{u2}^{-1} represent the diagonal matrix that normalizes the columns of $(D_1^p T)$. Then from Equations 24 and 25 Equations 27 and 28 follow as:

$$D_{u2}^2 = diagonal\ [T'D^{2p}T]; \qquad [27]$$

$$Z_u = D_{u2}^{-1} T' D^{2p} T D_{u2}^{-1}. \qquad [28]$$

Let the matrix $(D^p T D_{u2}^{-1})$ represent the transformation matrix for the primary structure matrix. The transpose of the inverse of this matrix is $(D^{-p} T D_{u2}^{1})$.

The matrix necessary to transform F to the primary pattern matrix, W_u, as determined by the u-th iterative stage is $(D^{-p} T D_{u2}^{1})$.

$$W_u = F D^{-p} T D_{u2}^{1} \qquad [30]$$

However, if reference is made to W_u^*, the parallel projections of the n variable vectors onto the r long primary vectors, then the entries being referred to are parallel projections within the metric of the initial fixed frame. That is to say, W_u^* is just W_u without the column rescaling.

$$W_u^* = F D^{-p} T \qquad [31]$$

Comparing Equation 31 with Equation 12 it becomes immediately evident that

$$W_u^* = F_u^* = F D^{-p} T. \qquad [32]$$

The Holzinger parallel projections of the n variable vectors onto the r long primary vectors are identical to the Thurstone perpendicular projections of the n variable vectors onto the r long reference vectors.

That is to say, when the metric of the initial factor solution is retained in place of either unit length reference vectors or unit length primary vectors the Holzinger pattern matrix and the Thurstone structure matrix are identical. (In earlier papers on the obliquimax $\dot{W}_u^*$ and F_u^* were referred to as one matrix, the "basic matrix.")

The above paragraph follows logically from an earlier paper presented by Harris and Knoell (1948) in which they derive a diagonal matrix that will rescale the columns of a Holzinger pattern matrix to those of a Thurstone structure matrix. The inverse of this matrix will rescale the columns of a Thurstone structure matrix to those of a Holzinger pattern matrix. Their discussion centered around the geometry of the two solutions and was considerably less complex, algebraically, than this section.

In their discussion Harris and Knoell (1948) demonstrate that the Thurstone structure values represent the bases and the Holzinger pattern values represent the hypotenuses of similar right-angled triangles. The elements of the diagonal rescaling matrix that they use for their conversion represent the correlations between the reference vectors and their associated primary vectors. They note that a primary vector is defined by the intersection of $(r - 1)$ hyperplanes and it is uncorrelated with the normals to these hyperplanes. Therefore, each primary vector is by definition orthogonal to all but one reference vector. Each reference vector is correlated with just one primary vector.

The correlation between a unit length reference vector and a unit length primary vector is defined as the scalar product of the paired direction cosines of the vectors. Equation 33 defines the diagonal matrix D_r whose ii-th element represents the correlation between the i-th reference vector and its associated i-th primary vector.

$$D_r = (D_{u1}^{-1}T'D^{-p})(D^{p}TD_{u2}^{-1}) = D_{u1}^{-1}D_{u2}^{-1} \qquad [33]$$

Summary of Section II

The basic framework for the general obliquimax has been discussed. The algebraic definitions of certain oblique solutions were discussed within the framework of common variance and perpendicular projections. The "solution equations" of the general obliquimax may now be summarized.

Reference Structure $= FD^{-p}TD_{u1}^{-1}$

Primary Pattern $= FD^{-p}TD_{u2}$

Primary Intercorrelations $= D_{u2}^{-1}T'D^{2p}TD_{u2}^{-1}$

Reference Intercorrelations $= D_{u1}^{-1}T'D^{-2p}TD_{u1}^{-1}$

Intercorrelations Between Primaries and Reference Vectors $= D_{u1}^{-1}D_{u2}^{-1}$

The discussion presented in this section is by no means complete. There has been a minimum of discussion concerning the exponent p and there has been no discussion with respect to the specific computation of T. In the next section the computation of T will be briefly discussed and the use of the exponent p will be somewhat clarified. This section has provided a basic rationale for the equations to be used in the simplified obliquimax.

Section III
The Simplified Obliquimax

Having discussed certain interpretative properties of the direction numbers and defined in part the analytic computations of the direction numbers it is now possible to re-establish the problem of computing an oblique transformation within an obliquimax framework. The iterative procedure developed in this section, referred to as the simplified obliquimax, is developed for pedagogical purposes. Certain matrices are

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modified in the iterative process to provide computational simplicity. The simplified obliquimax is a reliable semi-subjective transformation procedure that will allow beginning students to successfully compute an oblique transformation solution. It is also hoped that the simplified obliquimax will clarify certain aspects of oblique transformations in general.

Assume some factor loading matrix, F, defining the perpendicular projections of n variable vectors onto r mutually orthogonal factor axes. The r axes are arbitrarily orthogonal axes as determined by the initial factoring method. For illustrative and comparative purposes the simplified obliquimax will be discussed within the framework of Thurstone's (1947, pp. 140-144) classic box problem. The centroid solution for the box problem is reported in Table 1. For this particular set of data (n = 20) and (r = 3).

The r axes may be denoted as A_O, B_O and C_O. These arbitrary axes are regarded as fixed in position. The problem is to select by u successive approximations the unit reference vectors, A_u, B_u and C_u, such that the number of variable vectors with vanishing projections onto these unit reference vectors is a maximum.

With the exception of the exponent p and the final diagonal rescaling matrices all matrices necessary for computing either a reference structure or a primary pattern matrix can be determined prior to the computation of p. Thus, the problem at hand is one of determining a value for the exponent p such that the associated reference structure matrix will have a maximum number of vanishing projections.

In starting the iterations for the simplified obliquimax the matrix D_2^2 is determined from F through the use of Equation 9. Presented in Table 19 is D_2^2 whose ii-th element represents the total sum of the squared projections of the n variable vectors onto the i-th fixed axis.

$$D^2 = diagonal\ [F'F] \tag{9}$$

Table 19

The Positive Definite Diagonal Matrix Whose ii-th Element Represents the Variance Associated with the i-th Fixed Axis - Matrix D^2

A_o	B_o	C_o
12.567	.000	.000
.000	3.920	.000
.000	.000	3.154

The orthonormal transformation matrix T used in the simplified obliquimax is computed from F using Kaiser's (1958) normal varimax transformation procedure. A complete rationale for the use of the normal varimax transformation is beyond the scope of this paper. A very basic, though not particularly compelling, rationale for its use is that a primary vector may be defined in a special sense as a linear least squares approximation to a group of variable vectors without the restriction of mutual orthogonality and the normal varimax axes are with the restriction of mutual orthogonality in a special sense a linear least squares approximation to a group of variable vectors. It is assumed, and as will be seen, this is a critical assumption, that if the restriction of orthogonality were placed on the primary or reference vectors they would occupy the precise positions of the varimax axes.

Initially the matrix F must be transformed orthogonally to determine T. The matrix FT will be utilized in the simplified obliquimax as opposed to F. The matrix FT may be thought of as a preparatory matrix for the iterative stages (see Table 20 for *FT*). The varimax axes will be assumed as the initial r unit length reference vectors and will be referred to as A_v, B_v and C_v. (The subscript v referring to orthogonal varimax axes.) As in Thurstone's procedure the r unit length reference vectors are initially assumed to be mutually orthogonal. Contrary to Thurstone's procedure the initial unit reference vectors of the simplified obliquimax *are not* collinear with the r initial factor axes.

Table 20

Initial Factor Matrix Orthogonally
Transformed By Normal Varimax - Matrix *FT*

	A_v	B_v	C_v
1	0.052	-0.990	-0.109
2	0.146	-0.142	-0.973
3	0.984	-0.050	-0.103
4	0.135	-0.576	-0.803
5	0.887	-0.431	-0.117
6	0.872	-0.094	-0.472
7	0.091	-0.797	-0.605
8	0.443	-0.887	-0.141
9	0.566	-0.102	-0.820
10	0.118	-0.705	-0.710
11	0.692	-0.711	-0.134
12	0.733	-0.095	-0.681
13	0.046	-0.980	-0.139
14	0.200	-0.092	-0.954
15	0.958	-0.065	-0.060
16	0.790	-0.362	-0.462
17	0.384	-0.726	-0.568
18	0.057	-0.963	-0.071
19	0.097	-0.185	-0.940
20	0.975	-0.048	-0.112

Although graphical plots are not necessary in computing the simplified obliquimax they are presented initially for comparative and illustrative purposes. Figures 7, 8, and 9 represent the planar plots of the variable points with respect to the fixed axes. Also included in these figures are the projections of the unit reference vectors A_v, B_v and C_v.

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Figure 7

Planar Plots of Variable Vector Termini with Respect to Fixed Axes A_o' and B_o, Projected Onto Plane A_oB_o

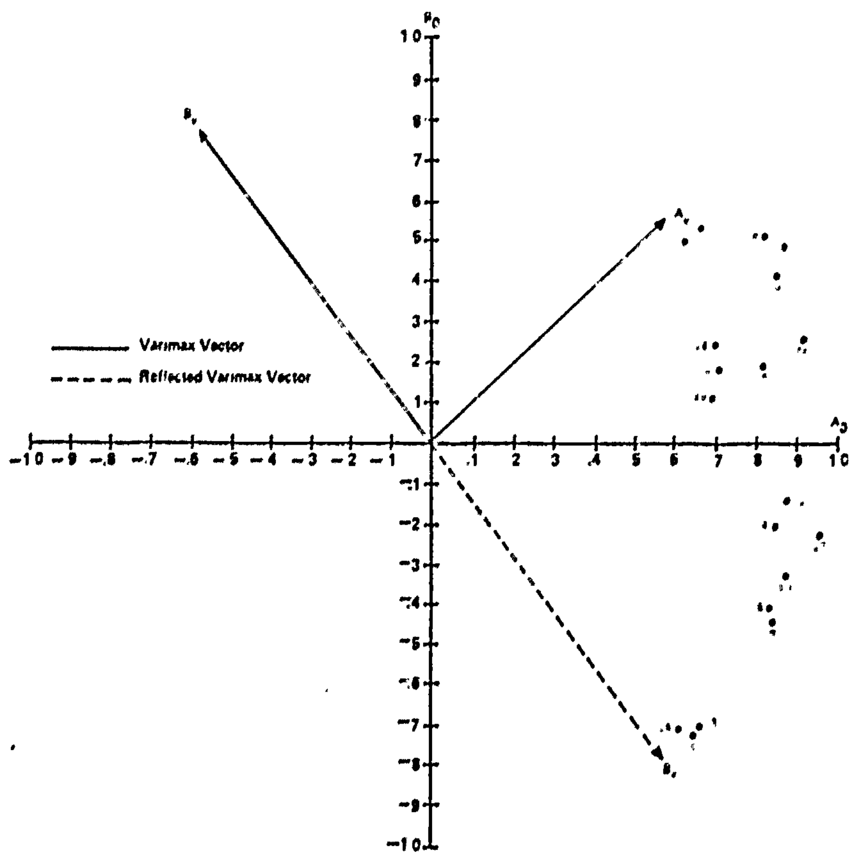


Figure 8

Planar Plots of Variable Vector Termini with Respect to Fixed Axes A_o and C_c, Projected Onto Plane A_oC_o

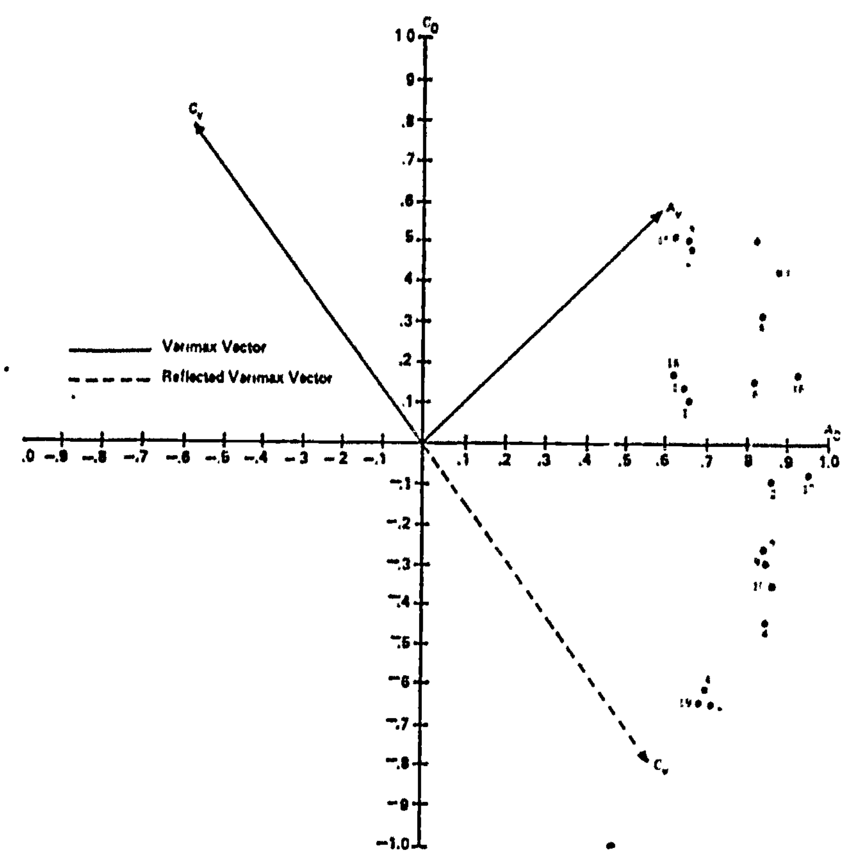
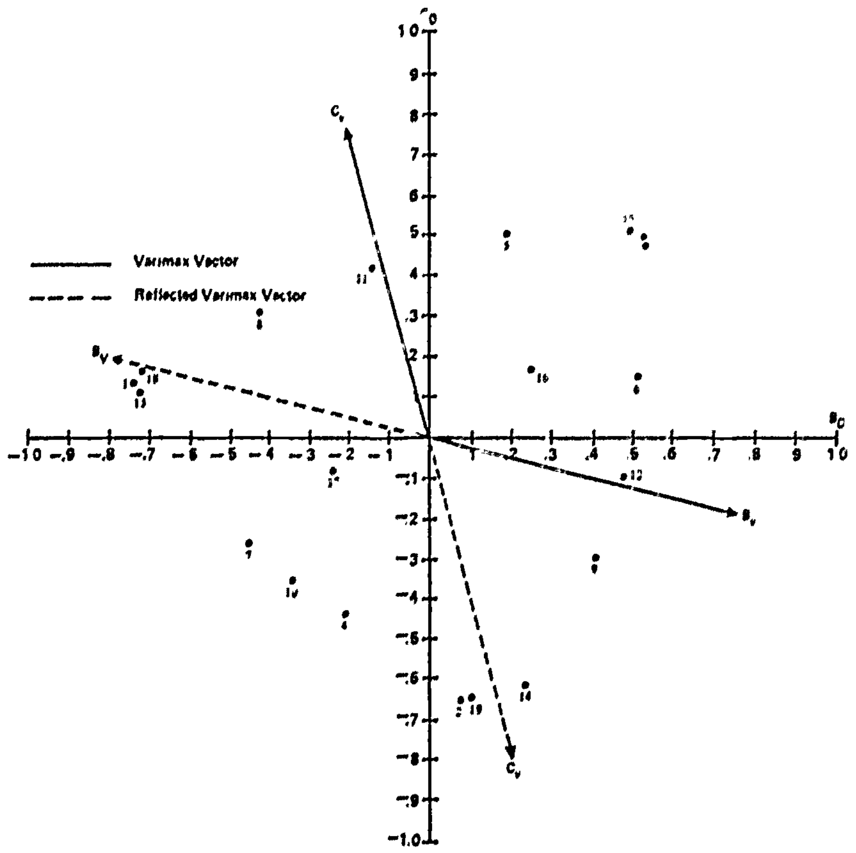


Figure 9

Planar Plots of Variable Vector Termini with Respect to Fixed Axes B_O and C_O, Projected Onto Plane $B_O C_O$



The normal varimax transformation matrix, T, is a matrix of direction cosines defining the unit reference vectors A_v, B_v and C_v with respect to the fixed axes A_o, B_o, C_o. This matrix will be utilized in all iterative stages and is presented in Table 21.

Table 21

The Orthonormal Transformation Matrix Computed From F By The Normal Varimax Transformation Procedure - Matrix T

	A_v	B_v	C_v
A_o	.587	-.572	-.574
B_o	.564	.797	-.217
C_o	.581	-.196	.790

In the previous section it was established that a good estimate of p will usually be some number bounded by zero and unity. Select some small number less than unity but greater than zero and call that number x. The initial estimate of the exponent p will be x. For the illustrative example x is chosen to be .20. (This choice for the illustrative example is not arbitrary. The rationale for choosing .20 instead of .05 or .50 will become apparent. In practice a choice of .10 would be a very safe estimate.)

Using x, T and D a matrix of direction numbers is defined as $T'D^{-x}T$. This matrix of direction numbers defines the termini of a new set of long reference vectors *with respect to the axes of the orthogonally transformed matrix* FT or the varimax axes A_v, B_v and C_v. Because the matrix T is orthonormal post-multiplying FT by $T'D^{-x}T$ will result in an Equation, 34, analogous to Equation 12.

$$F_u^* = FD^{-p}T \tag{12}$$

$$F_1^* = (FT)(T'D^{-x}T) = FD^{-x}T \tag{34}$$

Effectively, the post-multiplication of (FT) by $(T'D^{-x}T)$ is the same as post-multiplying (F) by $(D^{-p}T)$ if $p = x$. The matrix $(T'D^{-x}T)$ corresponds to the subjective Thurstonian matrix S_{12} at this point of the paper, however it will be referred to simply as S and for subsequent iterative stages it will be somewhat different in function than Thurstone's S_{mu} matrix.

$$S = T'D^{-x}T \tag{35}$$

In Figures 10, 11 and 12 the variable points have been plotted with respect to the columns of FT (see Table 20). The termini of the long reference vectors A_1', B_1' and C_1' have been plotted with respect to A_v, B_v and C_v, the varimax axes. The coordinates for the termini of the new long reference vectors with respect to varimax axes A_v, B_v and C_v (initial r unit length reference vectors) are given by the columns of the matrix S in Table 22.

Table 22

Symmetric Matrix of Direction Numbers of First Iterative Stage Reference Vectors with Respect To Previous Reference Vectors* - Matrix S

	A'_u	B'_u	C'_u
A'_{u-1}	.846	.030	.041
B'_{u-1}	.030	.842	-.034
C'_{u-1}	.041	-.034	.853

*Subscripts will be referred to in a later portion of this paper.

Figure 10

Planar Plots of Variable Vector Termini with Respect To Várimax Axes A_v and B_v, Projected Onto Plane A_vB_v

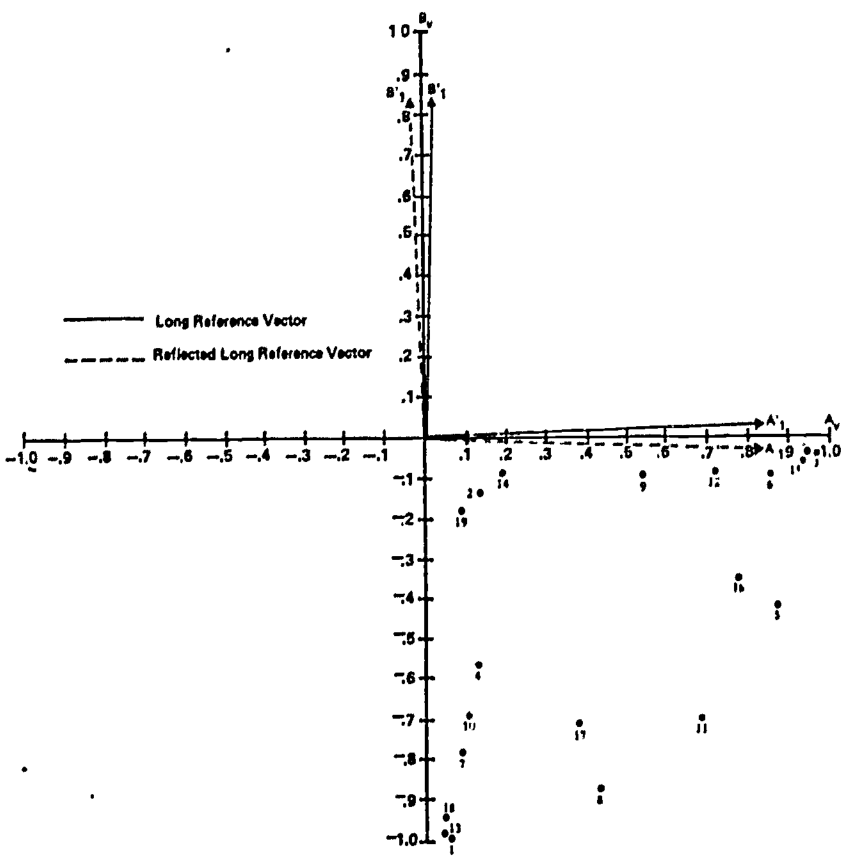


Figure 11

Planar Plots of Variable Vector Termini with Respect To Varimax Axes A_v and C_v, Projected Onto Plane $A_v C_v$

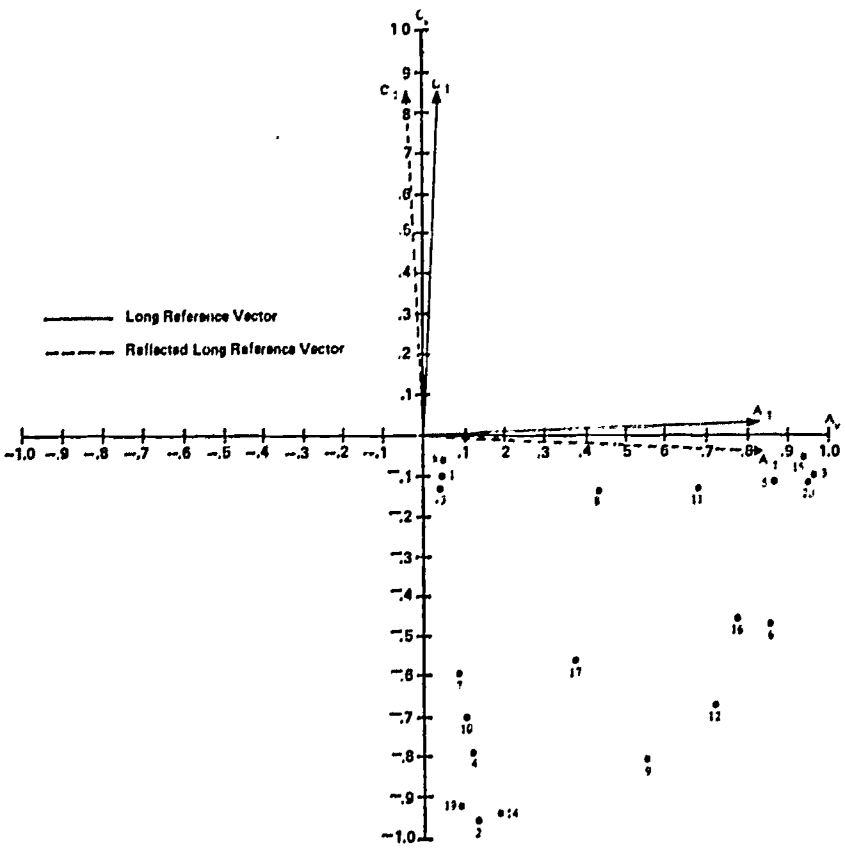
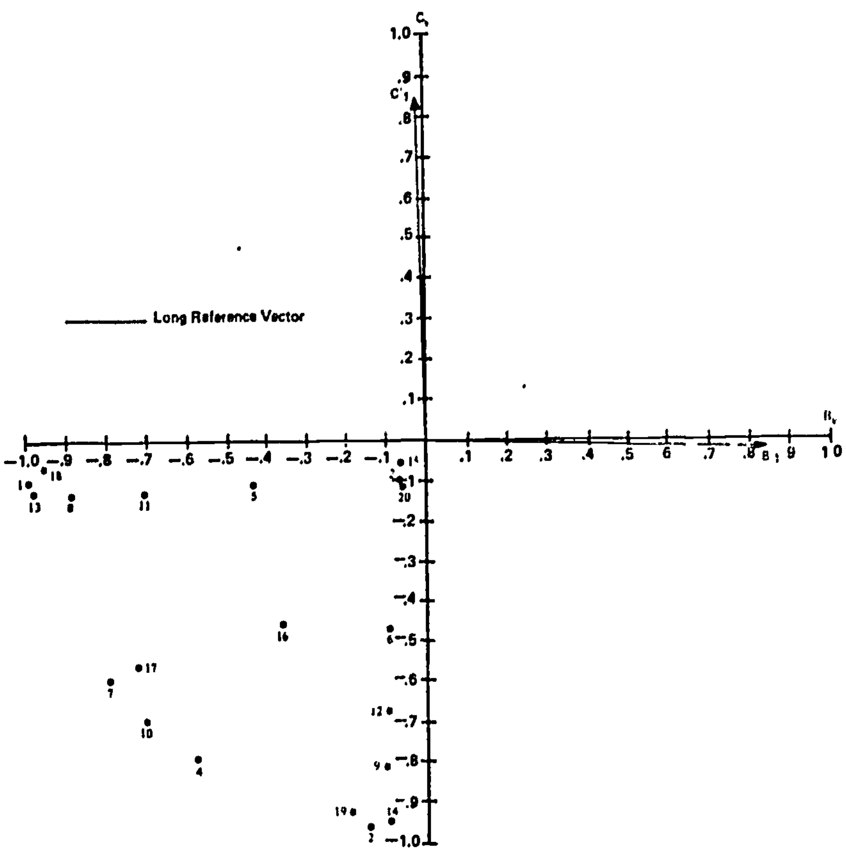


Figure 12

Planar Plots of Variable Vector Termini with Respect To Varimax Axes B_v and C_v, Projected Onto Plane $B_v C_v$



Equation 34 may be rewritten as Equation 36 to specifically define the long reference structure matrix with respect to the first iterative stage.*

$$F_1^* = FTS = FD^{-x}T \qquad [36]$$

The first iterative stage long reference structure matrix is reported in Table 23.

Table 23

Long Reference Structure Matrix as Determined By First Iterative Stage - Matrix F_1^*

	A_1'	B_1'	C_1'
1	0.010	-0.828	-0.057
2	0.079	-0.082	-0.819
3	0.826	-0.009	-0.046
4	0.064	-0.454	-0.659
5	0.733	-0.332	-0.048
6	0.715	-0.037	-0.364
7	0.028	-0.647	-0.485
8	0.343	-0.728	-0.072
9	0.442	-0.041	-0.673
10	0.050	-0.566	-0.577
11	0.559	-0.573	-0.061
12	0.589	-0.035	-0.547
13	0.004	-0.819	-0.083
14	0.127	-0.039	-0.802
15	0.806	-0.024	-0.010
16	0.638	-0.265	-0.349
17	0.280	-0.580	-0.443
18	0.017	-0.807	-0.025
19	0.038	-0.120	-0.791
20	0.819	-0.007	-0.054

Within Thurstone's algebraic framework the initial matrix of direction cosines, V_o, is post-multiplied by the matrix S_{o1} to form the matrix L_1. The matrix L_1 is the matrix of direction numbers for the long reference vectors with respect to the fixed frame. Within the obliquimax framework V_o is just T and S_{o1} is S, therefore:

*It is important to note here that unlike Thurstone (1947) we do not assume F to represent the first iterative stage of the oblique transformation solution, nor do we assume *FT* to represent the first iterative stage.

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$$L_1 = V_o S_{o1} = T(T'D^{-x}T) = D^{-x}T. \qquad [37]$$

If x is equal to p Equation 37 would be identical to Equation 10.

$$L_u = D^{-p}T \qquad [10]$$

The variance-covariance matrix of the first iterative stage reference vectors, Y_1^*, is computed using Equation 19 and may be inferred from Equation 5 and 37 as:

$$Y_1^* = T'D^{-2x}T = S^2 = SS = (T'D^{-x}T)(T'D^{-x}T) = S'S. \qquad [38]$$

Inasmuch as the matrix of direction numbers, S, is symmetric it is sufficient to simply multiply it by itself and note that $Y_1^* = S^2$.

Table 24

Variance Covariance Matrix of the First Iterative Stage Long Reference Vectors - Matrix Y_1^*

	A_1'	B_1'	C_1'
A_1'	0.713	0.049	0.069
B_1'	0.049	0.711	-0.057
C_1'	0.069	-0.057	0.730

The variance-covariance matrix, (see Table 25), of the first iterative stage long primary vectors, Z_1^*, is just the inverse of Y_1^*, (see Table 24), which is the product of the inverse of the direction numbers, S^{-1}, pre-multiplied by its transpose, $(S^{-1})'$. Inasmuch as S is a symmetric matrix Z_1^* is just S^{-2}.

$$Z_1^* = T'D^{2x}T = S^{-2} = S^{-1}S^{-1} = (T'D^{x}T)(T'D^{x}T) = (Y_1^*)^{-1} \qquad [39]$$

The problem of coalescing of the reference vectors has been eliminated by the definition of the obliquimax direction numbers. Therefore, unlike Thurstone's solution it is not necessary to rescale Y_1^* to form Y_1 to check for coalescing of reference vectors. At this point of the discussion it will be assumed that a second iteration is

Table 25

Variance Covariance Matrix of the First Iterative Stage Long Primary Vectors - Matrix Z_1^*

	A_1'-primary	B_1'-primary	C_1'-primary
A_1'-primary	1.416	-0.110	-0.140
B_1'-primary	-0.110	1.427	0.123
C_1'-primary	-0.142	0.123	1.395

needed. A discussion of the evaluation of the long reference vector structure matrix will be deferred until after the discussion of the computation of subsequent iterative stages. (For the reader's interest the first iterative stage unit length reference structure matrix and unit length reference vector intercorrelation matrix are reported in Tables 26 and 27 respectively.)

Table 26

Reference Structure Matrix as Determined By The First Iterative Stage - Matrix F_1

	A_1	B_1	C_1
1	0.012	-0.982	-0.067
2	0.093	-0.097	-0.958
3	0.975	-0.011	-0.054
4	0.075	-0.538	-0.772
5	0.865	-0.394	-0.056
6	0.844	-0.044	-0.426
7	0.033	-0.768	-0.568
8	0.404	-0.864	-0.084
9	0.522	-0.048	-0.787
10	0.059	-0.671	-0.675
11	0.659	-0.680	-0.071
12	0.695	-0.041	-0.640
13	0.004	-0.972	-0.097
14	0.150	-0.046	-0.938
15	0.951	-0.029	-0.011
16	0.753	-0.315	-0.409
17	0.330	-0.688	-0.519
18	0.020	-0.957	-0.029
19	0.045	-0.142	-0.926
20	0.967	-0.009	-0.063

Table 27

Intercorrelations of the Unit Length Reference Vectors As Determined By the First Iterative Stage - Matrix Y_1

	A_1	B_1	C_1
A_1	1.000	0.069	0.095
B_1	0.069	1.000	-0.079
C_1	0.095	-0.079	1.000

A symmetric matrix of direction numbers, S, was defined in the first iterative stage. This matrix of direction numbers will be a constant throughout all iterative stages. Specifically it describes the termini of the present iterative stage long reference vectors with respect to the previous stage long reference vectors. Here it must be emphasized that the obliquimax matrix of direction numbers S is distinctly different from Thurstone's matrix of direction numbers S_{mu}. For the first iterative stage these two matrices have identical roles. On subsequent iterative stages the obliquimax matrix S defines the termini of the present iterative stage long reference vectors with respect to the *previous iterative stage long reference vectors*. On subsequent iterative stages the Thurstone matrix S_{mu} defines the termini of the present iterative stage long reference vectors, u, with respect to the *previous iterative stage unit length reference vectors*, m. To compute the second iterative stage long reference structure matrix it is only necessary to post-multiply F_1^* by S. (See Table 28 for F_2^*; for continuity F_2 is reported in Table 29.)

$$F_2^* = FD^{-2x}T = (FD^{-x}T)(T'D^{-x}T) = F_1^*S \qquad [40]$$

Table 28

Long Reference Structure Matrix As Determined
By Second Iterative Stage - Matrix F_2^*

	A_2'	B_2'	C_2'
1	-0.019	-0.695	-0.020
2	0.031	-0.038	-0.692
3	0.697	0.019	-0.005
4	0.013	-0.357	-0.544
5	0.608	-0.256	0.000
6	0.589	0.003	-0.279
7	-0.015	-0.527	-0.390
8	0.265	-0.600	-0.022
9	0.345	0.002	-0.554
10	0.001	-0.455	-0.470
11	0.453	-0.463	-0.009
12	0.475	0.007	-0.441
13	-0.025	-0.686	-0.042
14	0.073	-0.001	-0.677
15	0.680	0.004	0.026
16	0.518	-0.192	-0.262
17	0.201	-0.465	-0.347
18	-0.011	-0.678	0.007
19	-0.004	-0.073	-0.669
20	0.690	0.020	-0.012

Table 29

Reference Structure Matrix As Determined By Second Iterative Stage - Matrix F_2

	A_2	B_2	C_2
1	-0.026	-0.972	-0.027
2	0.042	-0.054	-0.941
3	0.964	0.026	-0.007
4	0.018	-0.500	-0.740
5	0.841	-0.358	0.001
6	0.814	0.004	-0.380
7	-0.021	-0.738	-0.530
8	0.367	-0.840	-0.030
9	0.477	0.003	-0.753
10	0.002	-0.636	-0.639
11	0.626	-0.648	-0.013
12	0.657	0.010	-0.600
13	-0.034	-0.961	-0.058
14	0.101	-0.002	-0.921
15	0.941	0.006	0.035
16	0.716	-0.269	-0.357
17	0.278	-0.650	-0.471
18	-0.015	-0.948	0.010
19	-0.005	-0.102	-0.910
20	0.955	0.028	-0.016

Note that effectively ($TT' = I$) and that post-multiplying F_1^* by S is tantamount to rescaling the rows of $(D^{-x}T)$ by the matrix D^{-x} and then post-multiplying F by $(D^{-2x}T)$. The matrix $(D^{-2x}T)$ is simply the matrix of direction numbers defining the termini of the long reference vectors with respect to the initial fixed axes.

The variance covariance matrix of the second iterative stage long reference vectors is computed directly from Y_1^* through pre- and post-multiplication of Y_1^* by S. (See Table 30 for Y_2^*; for continuity Y_2 is reported in Table 31.)

Table 30

Variance Covariance Matrix of the Second Iterative Stage Long Reference Vectors - Matrix Y_2^*

	A_2'	B_2'	C_2'
A_2'	0.522	0.066	0.097
B_2'	0.066	0.511	-0.079
C_2'	0.097	-0.079	0.541

Table 31

Intercorrelations of the Unit Length Reference Vectors As Determined By The Second Iterative Stage - Matrix Y_2

	A_2	B_2	C_2
A_2	1.000	0.129	0.183
B_2	0.129	1.000	-0.150
C_2	0.182	-0.150	1.000

$$Y_2^* = T'D^{-4x}T = (T'D^{-x}T)(T'D^{-2x}T)(T'D^{-x}T) = SY_1^*S \qquad [41]$$

Equation 41 may be further simplified algebraically as:

$$Y_2^* = S^4. \qquad [42]$$

Following logically from Equation 42 is the computational equation for the variance covariance matrix of the second iterative stage long primary vectors. (See Table 32.)

$$Z_2 = S^{-4} \qquad [43]$$

From the discussion of the first two iterative stages of the simplified obliquimax it may be inferred that through an application of the laws of exponents any iterative stage can be developed directly from the first iterative stage without computing intermediate iterative stages.

Table 32

Variance Covariance Matrix of the Second Iterative Stage Long Primary Vectors - Matrix Z_2^*

	A_2'-primary	B_2'-primary	C_2'-primary
A_2'-primary	2.031	-0.327	-0.411
B_2'-primary	-0.327	2 057	0.359
C_2'-primary	-0.411	0.359	1.975

Assume that one wishes to determine the long reference structure matrix associated with the u-th iterative stage.

$$F_u^* = FT(T'D^{-x}T)_1\ (T'D^{-x}T)_2\ (T'D^{-x}T)_3 \ldots (T'D^{-x}T)_u$$
$$F_u^* = FT(T'D^{-x}T)^u = FTS^u$$
$$F_u^* = FT(T'D^{-ux}T) = FTS^u$$
$$F_u^* = FD^{-ux}T = FTS^u \qquad [44]$$

Because the matrix of direction numbers, S, is symmetric and because it is a constant at all iterative stages the matrix (FT) need only be post-multiplied by (S^u) to determine the u-th iterative stage long reference vectors. For the m-th iterative stage the matrix $F_{(m-1)}^*$ is simply post-multiplied by S to determine F_m^*.

The variance covariance matrix of the u-th iterative stage long reference vectors may be determined in a similar fashion. Assuming that the variance covariance matrix is computed by a pre- and post-multiplication of the previous iterative stage variance covariance matrix by S:

$$Y_u^* = (T'D^{-x}T)_u \ldots (T'D^{-x}T)_2\ (T'D^{-x}T)_1\ (T'D^{-x}T)_1\ (T'D^{-x}T)_2 \ldots (T'D^{-x}T)_u;$$
$$Y_u^* = (T'D^{-x}T)^u\ (T'D^{-x}T)^u;$$
$$Y_u^* = (T'D^{-x}T)^{2u} = S^{2u};$$
$$Y_u^* = (T'D^{-2ux}T) = S^{2u}. \qquad [45]$$

Thus, the variance covariance matrix of the u-th iterative stage long reference vectors is just the matrix of direction numbers raised to the $2u$-th exponential power. Following from Equations 42, 43 and 45 the variance covariance matrix of the u-th iterative stage long primary vectors is:

$$Z_u^* = (T'D^{2ux}T) = S^{-2u}. \qquad [46]$$

Using Equations 24 and 27 the diagonal rescaling matrices D_{u1}^2 and D_{u2}^2 may be determined from Y_u^* and Z_u^* respectively. The columns of matrix F_u^* as defined by Equation 44 may be rescaled by D_{u1}^{-1} to form the reference structure matrix, F_u, or they may be rescaled by D_{u2} to form the primary pattern matrix, W_u.

Table 33

Diagonal Rescaling Matrix Necessary to Rescale F_2^* to F_2 - Matrix D_{21}^{-1}

A_2'	B_2'	C_2'
1.383	.000	.000
.000	1.399	.000
.000	.000	1.360

Table 34

Diagonal Rescaling Matrix Necessary To Rescale F_2^* to W_2 - Matrix D_{22}

A_2'-primary	B_2'-primary	C_2'-primary
1.43	0.00	0.00
0.00	1.44	0.00
0.00	0.00	1.41

The matrices D_{21}^{-1} and D_{22}^{-1} are reported in Tables 33 and 34 respectively. The two column rescalings of F_2^* to form F_2 and W_2 are reported in Tables 29 and 35 respectively. The matrices Y_2^* and Z_2^* have been rescaled by D_{21}^{-1} and D_{22}^{-1} respectively to form Y_2 and Z_2 which are reported in Tables 31 and 36 respectively.

Table 35

Primary Pattern Matrix As Determined By Second Iterative Stage - Matrix W_2

	A_2-primary	B_2-primary	C_2-primary
1	-0.027	-1.001	-0.028
2	0.044	-0.055	-0.976
3	0.999	0.027	-0.007
4	0.019	-0.514	-0.767
5	0.869	-0.369	0.000
6	0.842	0.004	-0.393
7	-0.021	-0.756	-0.550
8	0.379	-0.864	-0.031
9	0.493	0.003	-0.781
10	0.001	-0.655	-0.662
11	0.648	-0.667	-0.013
12	0.679	0.010	-0.622
13	-0.036	-0.988	-0.059
14	0.104	0.001	-0.955
15	0.972	0.006	0.037
16	0.741	-0.276	-0.369
17	0.287	-0.670	-0.489
18	-0.016	-0.976	-0.010
19	-0.006	-0.105	-0.943
20	0.987	0.029	-0.017

Table 36

Intercorrelations of Unit Length Primary Vectors As Determined By The Second Iterative Stage - Matrix Z_2

	A_2-primary	B_2-primary	C_2-primary
A_2-primary	1.000	-0.159	-0.206
B_2-primary	-0.159	1.000	0.177
C_2-primary	-0.206	0.177	1.000

Comparing Tables 29 and 31 with Tables 13 and 14 of Thurstone's solution it is evident that the second iterative stage solution as determined by the simplified obliquimax closely corresponds to the final oblique solution as determined by Thurstone. For this illustrative example a reasonable estimate of p was (ux = .40). For any oblique solution the final estimate of p as determined by the u-th iterative stage will be ux.

In the simplified obliquimax the symmetric matrix of direction numbers, S, is used in an algebraically similar manner as Thurstone's matrix H_{mu} (Equations 6-8). Thurstone post-multiplied the m-th reference structure matrix by H_{mu} to obtain the u-th reference structure matrix. In the simplified obliquimax the (u-1)-th long reference structure matrix, (u - 1 = m), was post-multiplied by the symmetric matrix S to obtain the u-th long reference structure matrix. At each iterative stage Thurstone had to compute H_{mu}. In the simplified obliquimax the matrix S is a constant matrix that is computed prior to the iterative stages. If Thurstone had defined S_{mu} as a constant symmetric matrix and used it in place of H_{mu} his solution would be algebraically identical to the simplified obliquimax.

Additional Geometric Aspects of the Simplified Obliquimax Matrices S and $(D^{-p}T)$

Noting that the matrix S remains constant across iterative stages it may be correctly inferred that the i-th long reference vector of the u-th iterative stage has angles of inclination with the r long reference vectors of the (u-1)-th iterative stage that are *identical* to the angles of inclination that the i-th long reference vector of the (u-1)-th iterative stage has with the r long reference vectors of the (u-2)-th iterative stage. If in Figures 10, 11 and 12 the variable points were eliminated and the scalings of the axes were eliminated the remaining vectors would be representative of the plot of the new long reference vectors of the previous stage (see Figures 13, 14 and 15). It is prudent to keep in mind that even though the reference vectors are oblique, the plots are on orthogonal coordinate cross-section paper. As in Thurstone's approach (see *Subsequent Iterative Stages*) it is only for conceptual purposes that this plotting approach is used.

Figure 13

Long Reference Vectors A'_u and B'_u Plotted With Respect to Long Reference Vectors A'_{u-1} and B'_{u-1}, Projected Onto The $A'_{u-1}B'_{u-1}$ Plane

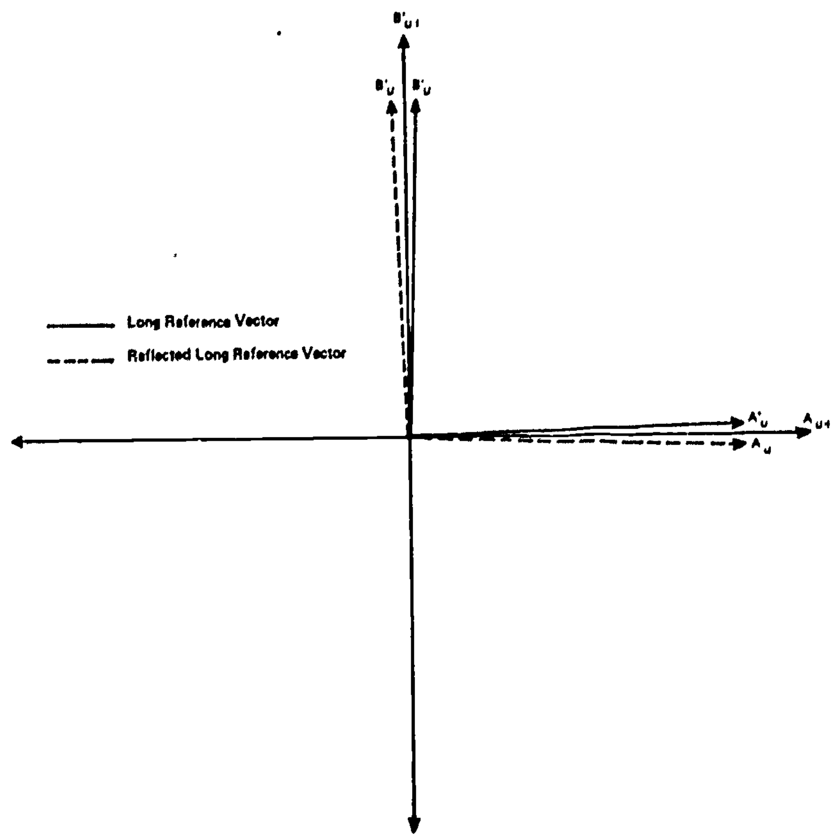


Figure 14

Long Reference Vectors A'_u and C'_u Plotted With Respect to Long Reference Vectors A'_{u-1} and C'_{u-1}, Projected Onto The $A'_{u-1}C'_{u-1}$ Plane

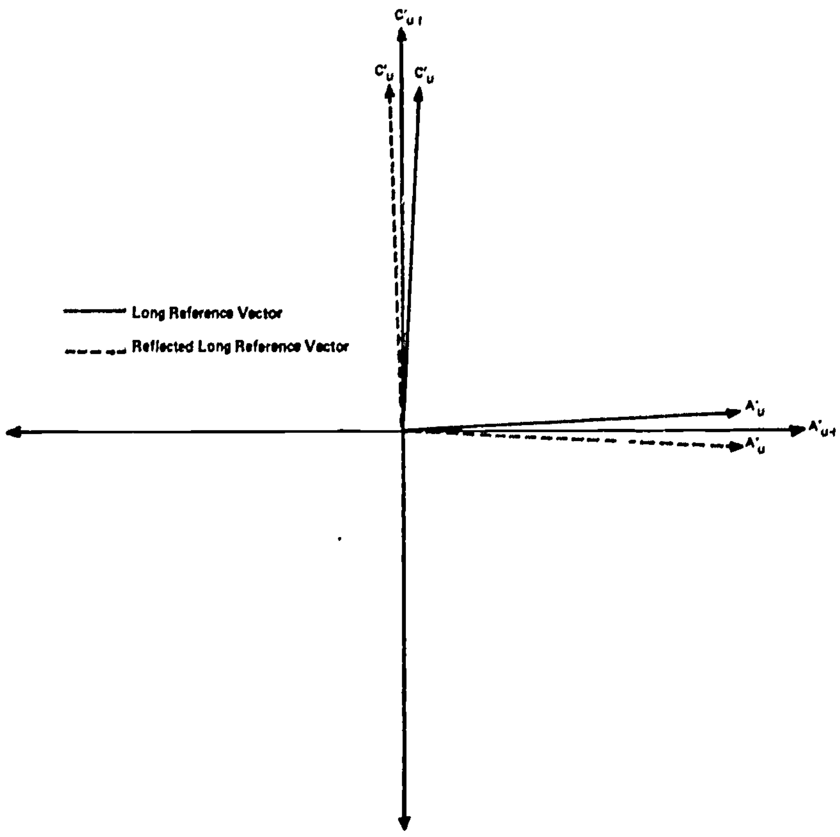
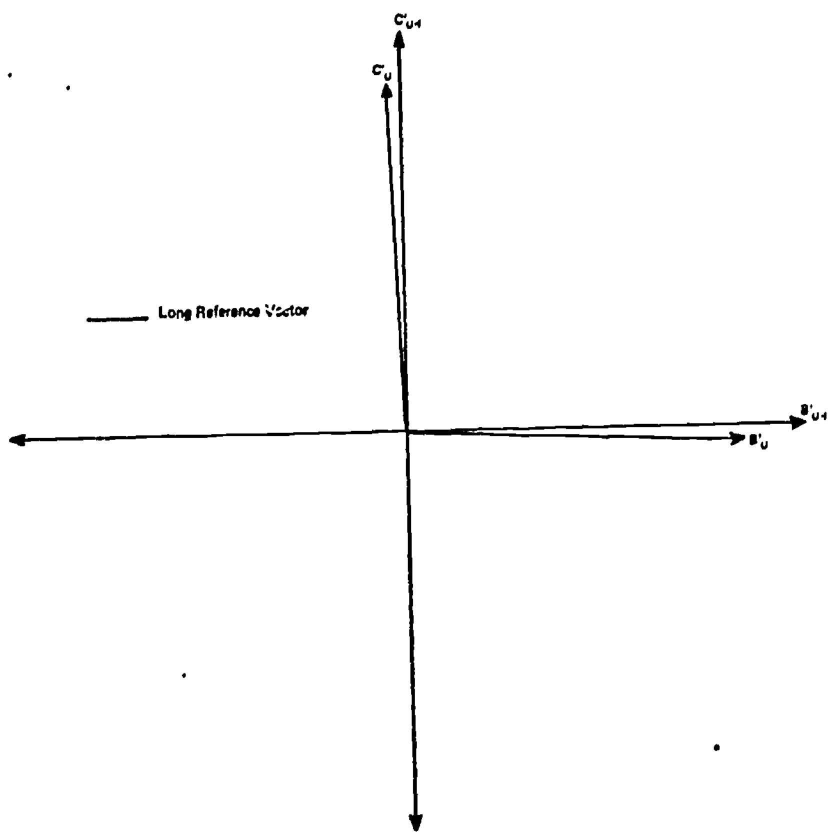


Figure 15

Long Reference Vectors B_u' and C_u' Plotted With Respect to Long Reference Vectors B_{u-1}' and C_{u-1}', Projected Onto The $B_{u-1}'C_{u-1}'$ Plane



In Figures 16, 17 and 18 the termini of the variable vectors are plotted with respect to the planes defined by the initial fixed axes. The unit reference vectors as determined through five successive iterative stages (x = .20) have also been plotted and labeled in these Figures. From these figures, several important geometric properties of the obliquimax may be inferred with respect to the matrices D^{-x}, D^{-p} and T.

The angle of inclination between the long reference vectors and the previous iterative stage reference vectors remains constant throughout the iterative stages. Although it is not clearly apparent in Figures 16, 17 and 18 there is a very systematic divergence of the reference vectors from the varimax axes toward the initial fixed axes from one iterative stage to the next iterative stage. If Figures 16, 17 and 18 were combined and plotted in a 3-dimensional space it would be clearly seen that the termini of the reference vectors follow distinct, *predictable* "paths" in space from one iterative stage to the next. Each reference vector will appear to converge toward the initial fixed axis associated with the smallest value in the matrix D^2. Within the framework of planar plots it may be observed that both reference vectors will tend to diverge from each other but in so doing they will converge toward that fixed axis, of the two defining the plane, that is associated with the smaller value in the matrix D^2.

Within the framework of the general obliquimax and, hence, the simplified obliquimax there are well defined paths in n-dimensional space that the termini of the reference vectors will follow through successive iterations. These paths are determined *a priori* by the orthonormal matrix T. An orthonormal matrix computed from F by the quartimax or equamax (assume r > 2) criteria will not define paths that are the same

Figure 16

Five Successive Approximations to Long Reference Vectors A' and B' Projected Onto The A_oB_o Plane Defined By The Original Fixed Axes A_o and B_o

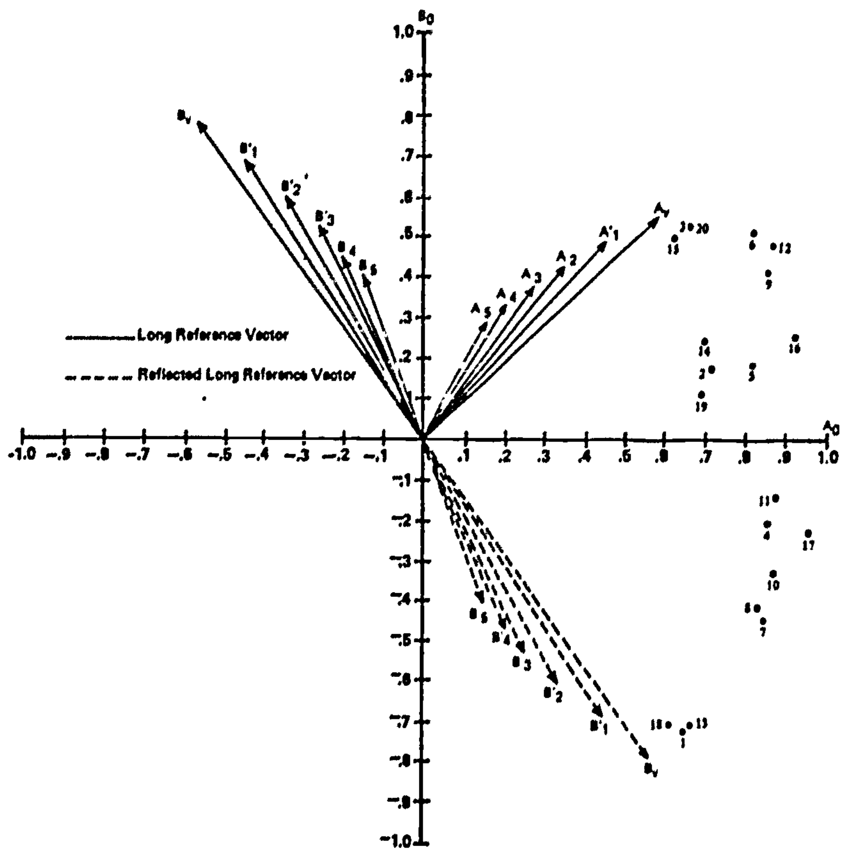


Figure 17

Five Successive Approximations to Long Reference Vectors A' and C' Projected Onto The A_oC_o Plane Defined By The Original Fixed Axes A_o and C_o

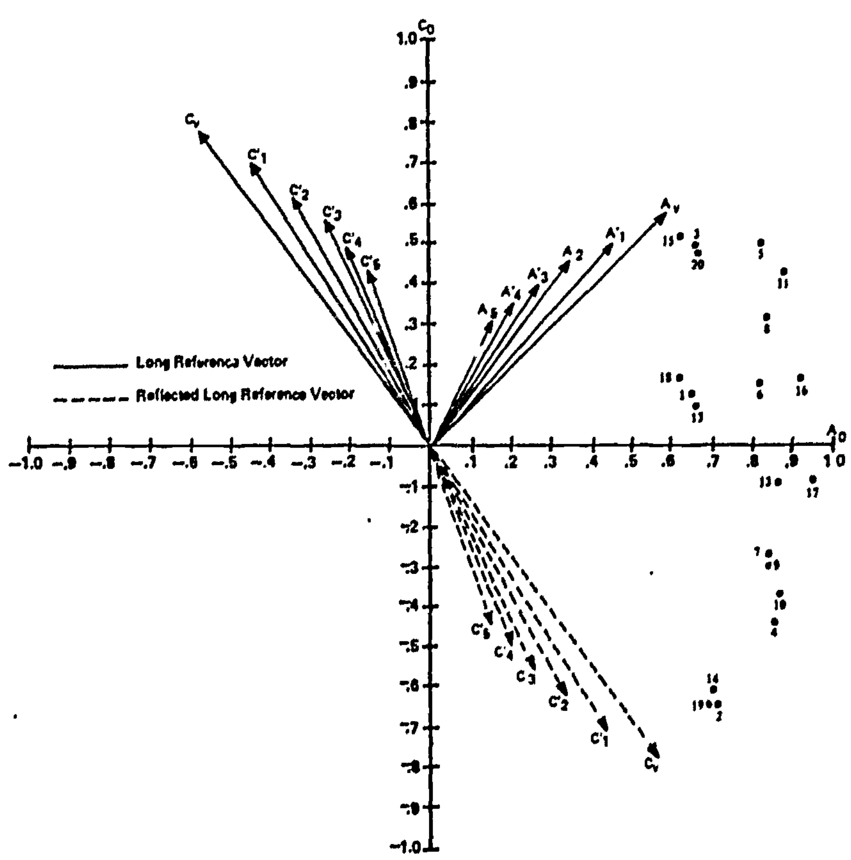
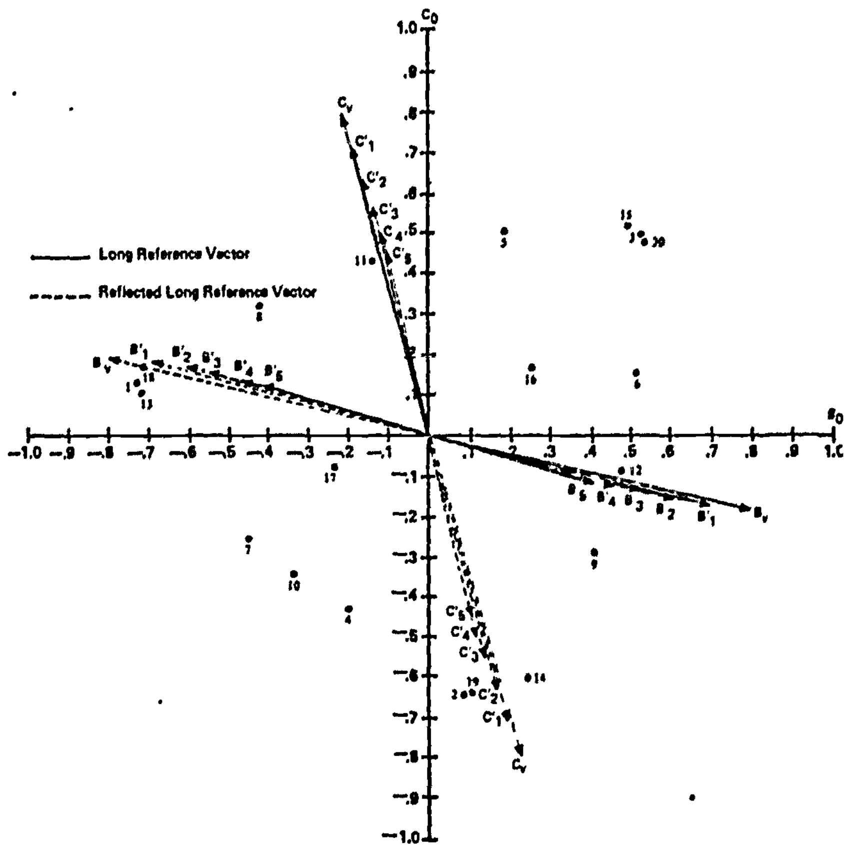


Figure 18

Five Successive Approximations to Long Reference Vectors B' and C' Projected Onto The B_oC_o Plane Defined By The Original Fixed Axes B_o and C_o



as the paths defined by the varimax criterion. Aside from the somewhat weak algebraic rationale for the varimax transformation matrix, two years of empirical investigation has firmly established the superiority of the varimax T in the general obliquimax transformation (The varimax T has never failed to give good results for a properly factored set of data.). Although these paths are not fully understood it is known that they meet at the origin. At the origin all of the reference vectors will coalesce into a point. As the termini of the reference vectors traverse these paths toward the origin the reference vectors become progressively more correlated. These paths may be discussed superficially as a group using the simple analogue of (rate x time = distance). The rate at which the reference vector termini traverse their paths as a group may be thought of as x. A unit of time may be thought of as an iterative stage. The total distance traversed is then a function of the value of x in the simplified obliquimax equations and the number of iterative stages.

It may be assumed that at some relative distance p along these paths there are r points defining the termini of r reference vectors such that the number of vanishing perpendicular projections of the variable vectors onto these r reference vectors is a maximum.

Assume that ($p \approx .40$), as it does in the illustrative problem. A choice of ($x = .01$) would denote a relatively slow rate requiring more time to arrive at distance ($p \approx .40$) than a rate of ($x = .20$). When ($x = .01$) the time required to arrive at p would be ($u = 40$) the number of iterative stages. When ($x = .20$) the time required to arrive at p would be ($u = 2$) the number of iterative stages.

Note that in Figures 16, 17 and 18 the reference vectors passed through p and that the number of vanishing projections onto the reference

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vectors determined by the fifth iterative stage is less than the number of vanishing projections onto the reference vectors determined by the second iterative stage. The reference vectors associated with (x = .25) may be inferred from Figures 16, 17 and 18. For (x = .25) the reference vectors will always, for any particular iterative stage, diverge farther than those reference vectors associated with (x = .20). The first iterative stage for (x = .25) would be a better approximation to (p = .40) than (x = .20), but on the second iterative stage those reference vectors associated with (x = .20) would be the reference vectors associated with (p = .40) while the second iterative stage reference vectors associated with (x = .25) would have diverged beyond the ideal reference vectors associated with (p = .40). Thus, it is most prudent that the estimate of x be *small*. (A reasonable estimate for most sets of data would be .10.) The larger the estimate the more probable is the possibility of jumping past p in a single iterative stage. A value of (x = .20) was chosen for the illustrative example as the ideal value of p was known to be approximately (.40) and a value of (x = .20) would only necessitate two iterative stages.

When to Stop Iterating

Having discussed the simplified obliquimax computationally and geometrically it is now necessary to briefly discuss the concept of vanishing projections within the obliquimax framework. Ideally one would like some analytic criterion to assess an oblique solution, however the discussion of such a criterion will not be presented in this paper. (Two analytic criterion have been established for use with the general obliquimax and an additional four are being investigated.)

Traditionally a subjective evaluation of the unit reference structure matrix loadings is used to judge the suitability of an oblique solution.

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If the small entries of the u-th iterative stage reference structure matrix are larger than their corresponding values in the $(u-1)$-th iterative stage reference structure matrix it may be assumed that for the set of possible solutions that might be defined using a specific value of x the $(u-1)$-th solution is the best in terms of the number of vanishing projections onto the unit length reference vectors.

Table 37

Long Reference Structure Matrix As Determined
By Third Iterative Stage - Matrix F_3^*

	A_3'	B_3'	C_3'
1	-0.038	-0.585	0.006
2	-0.004	-0.008	-0.588
3	0.589	0.037	0.024
4	-0.022	-0.281	-0.451
5	0.506	-0.197	0.034
6	0.486	0.030	-0.214
7	-0.045	-0.431	-0.315
8	0.205	-0.497	0.013
9	0.269	0.031	-0.458
10	-0.032	-0.367	-0.385
11	0.369	-0.376	0.027
12	0.383	0.035	-0.357
13	-0.043	-0.577	-0.014
14	0.034	0.024	-0.574
15	0.577	0.023	0.050
16	0.421	-0.137	-0.196
17	0.142	-0.373	-0.271
18	-0.029	-0.571	0.029
19	-0.033	-0.038	-0.568
20	0.584	0.038	0.018

Within the framework of the simplified obliquimax the long reference structure matrix may be evaluated in place of the unit length reference structure matrix. The number of vanishing projections onto the long reference vectors will also be a maximum for the iterative stage associated with the best estimate of p. In Tables 23, 28 and 37 the first three iterative stage long reference vector structure matrices are reported. In comparing Tables 23 and 28 the low loadings associated with Table 28 are smaller in magnitude than their corresponding loadings in Table 23. In comparing Tables 28 and 37 it may be observed that the low loadings in column three of Table 37 are smaller in magnitude than their corresponding loadings in column three of Table 28, however the low loadings in column one of Table 37 are larger in magnitude than their corresponding loadings in column one of Table 28. This phenomena of reduction and inflation of loadings in Table 37 may be thought of as "dumping" and actually represents the first stages of the coalescing of factors one and three. Inasmuch as all small loadings did not decrease in the third iterative stage long reference structure matrix it may be assumed that the long reference structure matrix determined by the second iterative stage is the best long reference structure matrix of the set of possible long reference structure matrices that might be determined for (x = .20). That is to say, the third and all subsequent iterative stages for (x = .20) will locate the termini of the reference vectors beyond the ideal points defined by p. (See Figures 16, 17 and 18.)

Summary of Section III

In this section the simplified obliquimax was developed and explained through the use of an illustrative example. A symmetric matrix of direction numbers, S, was used in the computation of the solution matrices at each iterative stage. The simplified obliquimax retained the metric of the original factor solution throughout all iterative stages and it was not until the final iterative stage solution had been obtained that the metric was converted to that of unit length reference vectors, thereby providing a traditional oblique solution.

The essence of the simplified obliquimax is the use of the symmetric matrix S and the retention of the metric of the original factor solution. The equations presented in this section are summarized below for the computation of the u-th iterative stage.

1. Define D^2 as:

   $$D^2 = diagonal\ [F'F]. \qquad [9]$$

2. Determine the orthonormal T from F using Kaiser's (1958) normal varimax procedure.

3. Select some small value for x, say:

   $x = .10.$

4. Define S as:

   $$S = T'D^{-x}T. \qquad [35]$$

5. The long reference structure matrix for the u-th iterative stage is defined as:

   $$F_u^* = FT(T'D^{-x}T)_1\ (T'D^{-x}T)_2 \ldots (T'D^{-x}T)_u.$$

   Therefore:

   $$F_u^* = FTS^u. \qquad [42]$$

6. The variance covariance matrix for the u-th iterative stage reference vectors is defined as:

   $$Y_u^* = (T'D^{-x}T)_u \ldots (T'D^{-x}T)_2\ (T'D^{-x}T)_1\ (T'D^{-x}T)_1\ (T'D^{-x}T)_2 \ldots (T'D^{-x}T)_u;$$
   $$Y_u^* = S^{2u}. \qquad [43]$$

7. The variance covariance matrix for the u-th iterative stage primary vectors is defined as:

   $$Z_u^* = (T'D^{x}T)_u \ldots (T'D^{x}T)_2\ (T'D^{x}T)_1\ (T'D^{x}T)_1\ (T'D^{x}T)_2 \ldots (T'D^{x}T)_u;$$
   $$Z_u^* = S^{-2u}. \qquad [44]$$

To convert to a traditional oblique solution it is only necessary to rescale F_u^*, Y_u^* and Z_u^* to the metric of unit vectors.

8. To determine the unit reference structure matrix define:

   $$D^2_{u1} = diagonal\ [S^{2u}] = diagonal\ [Y_u^*];$$
   $$F_u = F_u^* D_{u1}^{-1}.$$

9. To determine the unit primary pattern matrix define:

$$D_{u2}^{2} = \textit{diagonal}\ [S^{-2u}] = \textit{diagonal}\ [Z_u^*];$$

$$W_u = F_u^* D_{u2}^{+1}.$$

10. Using D_{u1}^{-1}, D_{u2}^{-1}, Y_u^* and Z_u^* the reference vector and primary vector intercorrelation matrices may be computed as:

$$Y_u = D_{u1}^{-1} S^{2u} D_{u1}^{-1} = D_{u1}^{-1} Y_u^* D_{u1}^{-1};$$

$$Z_u = D_{u2}^{-1} S^{-2u} D_{u2}^{-1} = D_{u2}^{-1} Z_u^* D_{u2}^{-1}.$$

All solution matrices within the framework utilize the symmetric matrix S. The definition of S eliminates the possibility of transforming to singularity and the necessity of planar plots.

Summary

This paper was arranged into three sections; the first section being primarily background; the second section being primarily theoretical; the third section being application and theory.

In the first section one of Thurstone's methods of determining oblique transformations was discussed within the context of his classical box problem. This section was presented to provide a background and to establish the terminology and methodology used in the subsequent sections.

In the second section of this paper certain theoretical aspects of the general obliquimax were discussed to provide a basis for the development and understanding of the simplified obliquimax. In this section a cursory discussion of the general obliquimax was provided. Total variance was defined with respect to the perpendicular projections of n variable vectors onto r *long* reference vectors and with respect to the perpendicular projections of r *long* primary vectors onto each other. The equations of the general obliquimax were discussed within the framework of variance and variance modification. This discussion necessitated an algebraic comparison of the Thurstone type reference structure matrix and the Holzinger type primary pattern matrix. Finally it was demonstrated algebraically that when the metric of the original factor solution is retained in place of unit vectors the Holzinger pattern matrix and the Thurstone structure matrix are identical.

In the third and final section of this paper the semi-subjective simplified obliquimax transformation was developed and discussed within the context of Thurstone's box problem. A constant, symmetric matrix of direction numbers was discussed algebraically and geometrically. The general obliquimax equations were modified and re-defined within the metric of the original factor solution using exponential powers of the symmetric matrix of direction numbers and an orthogonally transformed version of the initial factor loading matrix. Finally the subjective evaluation of the "simple structure" of a solution was discussed with respect to the un-rescaled reference structure matrix. The discussion was presented within the context of the box problem and in numerous parts of the section comparisons and contrasts were made with the Thurstonian model for determining oblique transformation solutions.

Conclusion

The primary objectives of this paper were pedagogical. One objective was to provide a reliable, semi-subjective transformation procedure that might be used without difficulty by beginning students in factor analysis. A second objective was to clarify and extend the existing knowledge of oblique transformations in general. A third objective was to provide a brief but meaningful explication of the general obliquimax. Implicit in these first three objectives was the fourth objective which was one of presenting a paper that might be profitable for both the beginning student and the factor analytic theoretician.

To these ends the simplified obliquimax was developed having as its basis the classic Harris and Kaiser (1964) theory of developing oblique transformation solutions through the use of orthogonal transformation matrices. Inasmuch as the Harris and Kaiser theory does encompass the Thurstonian approach the Thurstonian method of determining oblique transformations was used to provide background information and to explain by analogy certain aspects of the simplified obliquimax.*

It may be concluded that the simplified obliquimax has fulfilled the first three objectives of this paper. This paper has provided:

1. a reliable, semi-subjective transformation procedure for beginning students in factor analysis;

2. a clarification and extension of the existing knowledge of oblique transformations;

3. a brief explication of the general obliquimax.

*I would like to acknowledge the assistance that I received in this paper from J. Whitey. His discussions with me on the geometric aspects of oblique solutions proved quite valuable in the overall development of the total paper.

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